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Recent mathematical results in the nonlinear theory of flat and curved elastic membranes of revolution

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Abstract. The present article reviews some recent developments in nonlinear elastic membrane theory with special emphasis on axisymmetric deformation of flat circular and annular membranes subjected to a vertical surface load and with prescribed radial stresses or radial displacements at the edges. The nonlinear Föppl membrane theory of small finite deflections as well as a simplified version of Reissner's finite-rotation theory is employed, assuming linear stress-strain relations. The main analytical techniques are reported which have been applied recently in order to determine the ranges of those boundary parameters for which solutions of the relevant nonlinear boundary value problems exist, and ranges of parameters for which the principal stresses are nonnegative everywhere. Concerning plane membranes, it is shown how the mathematical theory of existence and uniqueness was nearly completed in recent works in contrast to curved membranes where references can be given to rather few results.

1. Introduction

In the linear theory of elastic membranes and thin shells, the deformation at any given place of the body is proportional to the magnitude of the applied load. As in three-dimensional elasticity, the load must be sufficiently small for linear membrane or shell theory to be applicable; otherwise a nonlinear theory is required. If the strain-displacement relations are nonlinear, but linear stress-strain relations are adequate, we have a *geometrically* nonlinear theory. When the latter relations are also nonlinear, one deals with a *physically* nonlinear theory. Many conventional materials for engineering thin shell and membrane designs are linearly elastic for small strain so that linear stress-strain relations are appropriate. Thus for sufficiently thin structures there is good reason to formulate geometrically nonlinear theories assuming finite deflections (rotations) but small strains, and to investigate the solution structure of the basic boundary value problems. This is the purpose of the present review, where we restrict ourselves to the class of problems described by nonlinear membrane theories. For simplicity, we only consider membranes of revolution under axisymmetric loads (aximembranes). Recently, materially nonlinear deformation has been studied for rubber-like membranes, but this topic is beyond the scope of this paper.

In 1859, Kirchhoff first proposed a nonlinear plate theory [28] which was reduced by von Kármán to a set of two simultaneous equations for the normal displacement w and a stress function f [26]. Somewhat earlier, A. Föppl had derived a nonlinear theory for flat membranes [13]. These are all geometrically nonlinear theories for *small finite deflections*, that is, the displacement components u and v tangential to the plane of the membrane or plate are assumed small compared to the normal displacement w; furthermore, only quadratic terms in w and in the change of angles of rotation are retained.

The von Kármán equations have been studied extensively by both mathematicians[1] and engineers because their simple structure makes them amenable to both theoretical and

[1] For a recent survey see P.G. Ciarlet [7].

numerical analysis. In cartesian coordinates, these equations are

$$D\Delta^2 w - [f, w] = p(x, y)\,, \qquad A\Delta^2 f + \frac{1}{2}[w, w] = 0\,, \tag{1.1}$$

where D and A are elasticity constants, f the Airy stress function, p is the surface load normal to the (undeformed) plate, Δ^2 is the biharmonic operator and $[f, g] = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}$. The Föppl membrane equations [13] are obtained by setting the flexural rigidity D equal to zero in (1.1).

In 1938, a nonlinear theory of shallow shells was given by Marguerre [31], a few years after Donnell [12] had derived a nonlinear theory for circular cylindrical shells to investigate certain buckling problems. Again these papers are restricted to a nonlinear theory with *small finite deflections*. The Marguerre equations are

$$\begin{aligned} &D\Delta^2 w - [z, f] - [w, f] = p(x, y)\,, \\ &A\Delta^2 f + [z, w] + \frac{1}{2}[w, w] = 0\,, \end{aligned} \tag{1.2}$$

where $z(x, y)$ represents the (undeformed) shape of the middle surface. Due to the presence of the (linear) curvature terms $[z, f]$ and $[z, w]$, the solution structure of (1.2) changes drastically from that of (1.1), including, in particular, buckling under normal pressure.

In a related development, the need for a stress analysis of thin inflatable sheets led Bromberg and Stoker to develop a geometrically nonlinear theory for aximembranes [5].

The first geometrically nonlinear shell theory not restricted to *small finite deflections* was given by E. Reissner [37]. This work concerns axisymmetric bending and stretching of shells of revolution (axishells). The basic equilibrium and compatibility equations and stress-strain relations were reduced essentially to a system of two coupled second order ordinary differential equations for the meridional angle of rotation Φ and a stress function F. Various simplifications of the system derived in [37] have been proposed in recent years. One by Reissner [38], others by Koiter [29] and by Libai and Simmonds [30]. Some of these simplifications are parallel to the linear theory, as far as neglecting terms involving Poisson's ratio ν is concerned, others are obtained by neglecting $O(\varepsilon)$ terms compared to unity, where ε is a measure for the strain in the shell. The simplified Reissner equations [30] can be written in the form

$$\begin{aligned} &r(\Phi - \varphi)'' + (\Phi - \varphi)'\cos\varphi = \frac{1}{r}\cos\Phi + \frac{1}{D}(F\sin\Phi - rV\cos\Phi)\,, \\ &rF'' + F'\cos\varphi - \frac{1}{r}F = \frac{1}{A}(\cos\Phi - \cos\varphi) + (r^2 p_H)' + \nu r p_s\,. \end{aligned} \tag{1.3}$$

In cylindrical coordinates (r, θ, z) the parametric representation of the surface of revolution is taken in the form $r = r(s)$, $z = z(s)$, where s is the arclength along a meridian of the surface. Primes denote differentiation with respect to s. Thus we have $r' = \cos\varphi$ and $z' = \sin\varphi$, where φ is the angle made by the meridional tangent with the base plane of the surface of revolution. The components of the surface load p are denoted by appropriate subscripts (H = horizontal, V = vertical; s, n meridional and normal to the deformed surface). We note the relations

$$p_H = p_s\cos\Phi + p_n\sin\Phi\,, \qquad p_V = p_s\sin\Phi - p_n\cos\Phi\,, \tag{1.4}$$

$F = rR$, R and V are the radial (horizontal) and axial stress resultants, respectively. Depending on the type of loading, the effect of the Poisson's ratio term in (1.3) may be significant for moderate to large values of the load. In (1.3) meridionally uniform shell properties have been assumed (constant thickness, isotropy and homogeneity of the material).

In order to solve the differential equations (1.3), boundary conditions consistent with the small strain assumption must be formulated. In many cases of practical interest, these boundary conditions are nonlinear, for example, if the horizontal displacement u is prescribed at the edge. In terms of Φ and F, one has

$$u = rA[F' + rp_H - \nu(r^{-1}F\cos\Phi + V\sin\Phi)]\,. \tag{1.5}$$

Very thin shells have negligible bending stiffness. Therefore, an important special case is *nonlinear membrane theory*, which is obtained from (1.3) by setting $D = 0$, which yields $rV\cos\Phi = F\sin\Phi$. Substituting this into the second equation of (1.3) we find the basic equation for *geometrically nonlinear aximembranes*

$$(rF')' - \frac{1}{r}F = \frac{1}{A}\left\{\frac{F}{[F^2 + (rV)^2]^{1/2}} - \cos\varphi\right\} + (r^2 p_H)' + \nu r p_s\,. \tag{1.6}$$

The equations of the approximate *small finite deflection* theory can now be obtained as follows. Introducing $\beta = \varphi - \Phi$, one may write $\cos\Phi = \cos\varphi + \beta\sin\varphi - \frac{1}{2}\beta^2\cos\varphi + \cdots$ and a similar expansion for $\sin\Phi$. Retaining only terms up to the second degree in β and F in equations (1.3), we get the equations of small finite deflection theory. It is a simple exercise to transform equations (1.2) to cylindrical coordinates. Set $z = z(r)$, $w = w(r)$, $f = f(r)$ (axishells) and integrate the resulting equations twice with respect to r. The result will be the small finite deflection version of equations (1.3) for shallow shells of revolution. The *small finite deflection membrane equation* can be obtained directly from (1.6) by expanding the square root and retaining only quadratic terms. What comes out is a generalization of the Föppl membrane theory to curved membranes:

$$(rF')' - \frac{1}{r}F = \frac{1}{A}\left[1 - \cos\varphi - \frac{1}{2}\left(\frac{rV}{F}\right)^2\right] + (r^2 p_H)' + \nu r p_s\,. \tag{1.7}$$

For the special case of a plane membrane $z = \varphi = 0$ under vertical load $p_H = p_s = 0$, we get

$$(rF')' - \frac{1}{r}F = -\frac{1}{2A}\left(\frac{rV}{F}\right)^2. \tag{1.8}$$

This review paper is organized as follows. The mathematical theory for the boundary value problems of plane membranes is nearly complete, while for curved membranes relatively few results are available. Accordingly, Sections 2, 3, and 4 are devoted to circular and annular membranes. In the first part of Section 2, results of the small finite deflection theory (1.7), called the Föppl theory, are presented for circular membranes. In the second part, results for a (simplified) geometrically nonlinear theory involving arbitrary finite rotations (1.6), called the Reissner theory, are discussed for circular membranes. Similarly, annular membranes for the Föppl and Reissner theories are discussed in Section 3. The tensile solutions of Sections 2 and 3 are characterized by $\sigma_r \geqslant 0$ where σ_r is the radial stress component. In Section 4, the additional condition that the circumferential stress σ_θ satisfies $\sigma_\theta \geqslant 0$ is imposed, leading to what we call wrinkle-free solutions. Again these solutions are discussed both within the framework of the Föppl and the Reissner theories. Finally, some results for curved membranes are presented in Section 5.

2. Circular membranes

Consider a circular membrane of radius a and thickness d subjected to a vertical pressure $p = p(r)$. At the edge $r = a$, either a constant radial stress σ_r or a constant radial displacement u is prescribed. Assuming *small finite deflections*, we have the governing equation (1.8). Here $s = r$ and $F = r\sigma_r d$. Dimensionless variables are introduced by $r = ax$, $\sigma_r/E = k^2y/4$, $p(r) = p_0\bar{p}(x)$, where E is Young's modulus, related to A in (1.8) by $AEd = 1$, $p_0 > 0$ is the maximum of $|p(r)|$, so that $|\bar{p}(x)| \leqslant 1$, and $k = (2p_0a/Ed)^{1/3}$. $\bar{p}(x)$ is assumed piecewise continuous for $0 \leqslant x \leqslant 1$. In terms of x, y and $\bar{p}$, equation (1.8) can be reduced to the form

$$Ly := -y'' - \frac{3}{x}\, y' = \frac{2}{y^2}\, Q^2(x)\,, \qquad 0 < x < 1\,, \qquad Q(x) := \frac{2}{x^2}\int_0^x t\bar{p}(t)\,\mathrm{d}t\,. \tag{2.1}$$

Henceforth primes denote differentiation with respect to x, unless stated otherwise. If the surface load is *uniform*, then $\bar{p} = 1$ and therefore $Q = 1$. The circumferential stress σ_θ, the radial and normal displacements u and w are related to the variables x and y through

$$\begin{aligned} &\sigma_\theta/E = k^2(xy' + y)/4\,, \qquad u = ak^2x[xy' + (1-\nu)y]/4\,, \\ &w = ak\int_x^1 tQ(t)[y(t)]^{-1}\,\mathrm{d}t\,, \qquad 0 \leqslant \nu \leqslant 1/2\,. \end{aligned} \tag{2.2}$$

Solutions of (2.1) are sought satisfying the boundary conditions

$$y'(0) = 0\,, \qquad y(1) = S > 0 \quad \text{or} \quad y'(0) = 0\,, \qquad y'(1) + (1-\nu)y(1) = H \in \mathbb{R}\,, \tag{2.3}$$

depending on whether σ_r or u is prescribed at the edge $r = a$. Hence, (2.1) and (2.3) define two different boundary value problems for the circular membrane, called Problem S and Problem H in what follows. An integration of (2.1), making use of $x^3Ly = -(x^3y')'$, shows that

$$x^3y'(x) = -\int_0^x \frac{2t^3}{y^2(t)}\, Q^2(t)\,\mathrm{d}t \leqslant 0\,. \tag{2.4}$$

Accordingly, the solutions of both Problems S and H are monotone decreasing in the interval [0, 1]. In particular, we have $y(x) \geqslant S > 0$ in Problem S.

The first solution of (2.1, 2.3) for uniform pressure ($Q = 1$) and $H = 0$ was apparently given by Hencky [23], using *formal power series* in x. In a formulation slightly different from [23], we introduce $z = 2/y$ and seek a solution in the form

$$y(x) = \sum_{n=0}^{\infty} y_n x^{2n}\,, \qquad z(x) = \sum_{n=0}^{\infty} z_n x^{2n}\,, \qquad z_0 = 2/y_0\,. \tag{2.5}$$

The *convergence* of this series solution was first established in 1965 by Weinitschke [46], yielding first existence theorems for Problem S and Problem H for $H = 0$. A different and simpler convergence proof for both fixed edge ($H = 0$) and loaded edge (Problem S) was published later in [47], where uniqueness of solutions was also established.

The first step is to introduce the series (2.5) into $Ly = z^2/2$ and $yz = 2$, obtain recurrence relations for the coefficients y_n, z_n and show by induction that for suitable choices of A, B and $q > 1$

$$|y_n| \leq \frac{A}{(n+1)^{q+1}}, \qquad z_n \leq \frac{B}{(n+1)^q}, \quad n = 1, 2, \ldots . \tag{2.6}$$

This has the immediate consequence that the series (2.5) and the differentiated series $y'(x)$ are all uniformly convergent for $|x| \leq 1$. The next step is to show that there exist suitable choices of A, B and q such that the boundary conditions for Problems S and H, respectively, are satisfied. In Problem S it is easy to see that convergence will occur if S is sufficiently large. In Problem H, the boundary term $y' + (1 - \nu)y$ at $x = 1$ must be shown to change sign for two different choices of A, B and q. The details are quite technical, so we refer to [47] and [44]. The results can be summarized as follows:

THEOREM 2.1. *Problem S for $Q = 1$ and $S > S_0 = 4/5$ has a unique solution $y(x) > 0$, which admits a uniformly convergent power series expansion for $|x| \leq 1$, with $S < y_0 \leq S + (1/2S)^2$.*

THEOREM 2.2. *Problem H for $Q = 1$ and $H = 0$ has a unique solution $y(x)$. The solution is positive and can be represented by a power series, uniformly convergent for $|x| \leq 1$, with* $20/19 < y_0 < 2$ for $\nu = 1/3$.

The series solutions may be used for getting accurate numerical approximations to Problems S and H. A number of different numerical treatments of both circular and annular membrane problems have appeared in the engineering literature (see [25] and the references given there).

A different and more elegant existence proof for solutions of Problems S and H can be obtained by an *integral equation method*. This was first done by Dickey [10], again for the case of uniform pressure $Q = 1$. He showed existence of a unique $C^1[0, 1]$-solution for Problem S provided $S^3 > 4j^{-2}$, where j denotes the first zero of the Bessel function J_1, which amounts to $S > S_0 \doteq 0.648$, thus improving on Theorem 2.1. By an interpolation between values of S well above S_0, he also showed existence of a unique solution of Problem H for $H = 0$. The method will be discussed below.

In order to remove the above restrictions $Q = 1$ and $S > S_0$, Callegari and Reiss [6] employed a *shooting technique*. In this way, they first proved existence and uniqueness of solutions of Problem S for *all* $S > 0$, and for *variable load* $p(r)$. The proofs in [6] are quite technical and will not be described here. They are much more complicated than the ones given below by the integral equation method. Although the shooting method is constructive with respect to the initial value problem, no converging iteration scheme was given in [6] to solve the relevant boundary value problems.

A simple and standard way to find an *integral equation* for Problems S and H is to calculate the Green's functions $G(x, t)$ for the problems $x^3Ly = 0$ with homogeneous boundary conditions (2.3). It follows that the solutions of Problem S and Problem H, respectively, are equivalent to an integral equation [49]

$$y(x) = q + \int_0^1 \frac{2t^3}{y^2(t)} Q^2(t) G(x, t)\, dt =: Ty , \tag{2.7}$$

where $q = S$ for Problem S and $q = H/(1 - \nu)$ for Problem H, and

$$G(x, t) = \begin{cases} (x^{-2} + m)/2 , & 0 \leqslant t \leqslant x \leqslant 1 , \\ (t^{-2} + m)/2 , & 0 \leqslant x \leqslant t \leqslant 1 , \end{cases} \tag{2.8}$$

where $m = -1$ or $m = (1 + \nu)/(1 - \nu)$ for Problem S or Problem H, respectively. In the case $Q = 1$ the integral equation (2.7) for Problem S reduces to that obtained by Dickey [10].

With a view towards getting numerical solutions for Problems S and H, we first consider a constructive proof of the existence of solutions of (2.7). Starting with $y_0 = q > 0$, we define $y_n(x)$ by $y_{n+1} = Ty_n$.

LEMMA 2.3. *The operator T is* antitone: *If $0 < y \leqslant z$, then $Ty \geqslant Tz$.*

A general theorem on antitone operators (e.g., see [9]) implies that the sequence y_n has the following basic property.

LEMMA 2.4. *If $0 < y_0 \leqslant y_2 \leqslant y_1$ then for any positive integer n*

$$y_0 \leqslant y_2 \leqslant y_4 \leqslant \cdots \leqslant y_{2n} \leqslant y_{2n+1} \leqslant \cdots \leqslant y_3 \leqslant y_1 . \tag{2.9}$$

In view of Lemma 2.3 and the choice of y_0 the condition $y_0 \leqslant y_2 \leqslant y_1$ is satisfied for any $S > 0$ or any $H > 0$. In order to apply the Banach fixed point theorem we introduce a norm

$$\|y\| = \sup_{0 \leqslant x \leqslant 1} \{W(x)^{-1} |y(x)|\} , \quad W(x) > 0 .$$

The objective then is to find an optimal $W(x)$ such that T is contractive for a maximal range of the parameters S and H. The result of this calculation is [49]:

THEOREM 2.5. *Assume $Q^2(x) \leqslant c$, then the integral equation* (2.7) *has a unique solution for all $S > S_0 = 0.648c^{1/3}$ in Problem S and for all $H > H_0 = H_0(\nu)c^{1/3}$ in Problem H, where $0.648 < H_0(\nu) < 1.057$ for $1/2 \geqslant \nu \geqslant 0$. In these ranges of S and H, $y_n(x)$ converges uniformly to the solution $y(x)$. The convergence is alternating and $y(x) \in M_n$ where*

$$M_n := [y_{2n}, y_{2n+1}] := \{y(x) \mid y_{2n} \leqslant y \leqslant y_{2n+1}\} . \tag{2.10}$$

In particular, for $n = 0$, we have for the solution of Problem S

$$\begin{aligned} &S \leqslant y(x) \leqslant S + \frac{1}{S^2} \int_0^1 2t^3 Q^2(t) G(x, t)\, dt = y_1 , \\ &S \leqslant y(x) \leqslant S + \frac{1}{4S^2} (1 - x^2) , \quad \text{for } Q = 1 . \end{aligned} \tag{2.11}$$

The restrictions on the size of S and H can now be removed by applying Schauder's fixed point theorem. In view of Lemma 2.4, the operators T map the convex set M_n into itself, that is, for any positive integer n we have $TM_n \subseteq M_n$, provided $S > 0$ $(H > 0)$. T is completely continuous on the set $M_0 \supseteq M_n$ (e.g., see [9]). Hence the Schauder fixed point theorem is applicable and yields [49]:

THEOREM 2.6. *Problem S (Problem H) has at least one solution $y(x) > 0$ for* all $S > 0$ $(H > 0)$. *The solution is contained in M_n.*

The uniqueness is proved by a classical argument. Let y_1, y_2 be two positive solutions of (2.1), (2.3) then $w = y_1 - y_2$ satisfies

$$Lw = -M(x)w\,, \qquad M(x) = \frac{2Q^2(x)}{y_1^2 y_2^2}(y_1 + y_2) \geq 0\,, \tag{2.12}$$

$$w'(0) = 0 \quad \text{and} \quad w(1) = 0 \quad \text{or} \quad w'(1) + (1 - \nu)w(1) = 0\,. \tag{2.13}$$

By the maximum principle (see Protter and Weinberger [36]) it follows that $w \equiv 0$.

THEOREM 2.7. *Any positive solution of Problem S (Problem H) is* unique, *if $\bar{p}(x)$ is piece-wise continuous.*

The unique solutions of Problems S and H guaranteed by Theorems 2.6 and 2.7 cannot be constructed by a convergent iteration, yet we obtain upper and lower bounds to the exact solution by iteration.

It should be mentioned that the existence part of Theorem 2.6 was obtained independently by Stuart [43], as an application of his general theory of integral equations with decreasing nonlinearities. He also obtained convergence of the sequence $y_n(x)$ from Problem S in the range $S > S_0 = 0.648c^{1/3}$ (see Theorem 2.5).

It is worth noting that $S > 0$ covers the complete physically meaningful range of tensile solutions for Problem S, while $H > 0$ does not. In the latter case, a physically significant situation is $H = 0$. On the other hand, the condition $q > 0$ in (2.7) is essential for starting the iteration. If $H = 0$, the operator T is still antitone, but it does not seem possible to choose y_0 and y_1 such that $y_0 \leq y_2 \leq y_1$. T is not contractive, nor is the Schauder theorem applicable. Hence it appears that Problem H for $H = 0$ cannot be solved directly via an integral equation method. However, there is a simple idea, by which this case can be covered. Any solution $y(x; S)$ of Problem S is also a solution of Problem H for a value of H given by

$$N(S) := y'(1; S) + (1 - \nu)S = H\,. \tag{2.14}$$

A simple estimate yields $y'(1; S) \to -\infty$ as $S \to 0$ [49]. Since $N(S) \to +\infty$ as $S \to \infty$, there exists for any given real value of H a corresponding value $S > 0$ such that $N(S) = H$. This result, together with Theorem (2.7), yields

THEOREM 2.8. *Problem H has a unique positive solution for all real H.*

A substantial improvement of the convergence properties of the iteration $y_{n+1} = Ty_n$ can be achieved by an interpolated (averaged) iteration of the type

$$y_{n+1} = \alpha T y_n + (1 - \alpha) y_n . \tag{2.15}$$

This was first pointed out by Ostrowski [34]. It was rediscovered in connection with an elastic plate problem by Keller and Reiss [27], who found numerically, that the range of convergence was drastically increased by an appropriate choice of α. But the convergence has remained an open problem until Novak [33] recently proved the convergence of the iteration (2.15) for a class of operators T which includes the operator T defined in (2.7). More precisely, he proved the following

THEOREM 2.9. *Problem S has a unique solution $y(x)$ for all $S > 0$ which can be obtained as the limit of the iteration* (2.15). *The interpolated iteration converges for all $\alpha > 0$ which are sufficiently small.*

Hence we have arrived at a constructive method to solve Problem S in the full range $S > 0$. In addition it turns out that the iteration (2.15) also converges for the operator T corresponding to Problem H *including* the important case $H = 0$ [49]. Numerical results based on (2.15) are also given in [49]. The value of S in (2.14) corresponding to the fixed edge problem $H = 0$ is approximately $S = 0.8549$ (for $\nu = 1/3$).

Next we turn to the boundary value problems posed by Reissner's *finite rotation theory*. The governing equation is (1.6), which can also be derived directly (without reference to shell theory), as shown by Clark and Narayanaswamy [8]. Furthermore, the equations in [8] can be shown to be equivalent to those derived earlier by Bromberg and Stoker [5]. We consider again a circular membrane under vertical pressure $p_V = p(r)$, with prescribed σ_r (Problem S) or prescribed u (Problem H) at the edge $r = a$. Hence $p_H = 0$ and $p_s = p_V \sin \Phi$ in equation (1.6). Introducing the same dimensionless variables as before, we obtain the basic differential equation and the boundary conditions for Problems S and H,

$$\begin{aligned} &Ly = f(x, y) - 2\nu k^2 \bar{p}(x) Q(x) / D(x, y) , \quad 0 < x < 1 , \\ &f(x, y) := (2/kx)^2 [1 - y/D(x, y)] , \qquad D(x, y) := [y^2 + k^2 x^2 Q^2(x)]^{1/2} , \\ &y'(0) = 0 \quad \text{and} \quad y(1) = S \quad \text{or} \quad y'(1) + y(1) - \nu D(1, y(1)) = H . \end{aligned} \tag{2.16}$$

The meridional and circumferential (Piola–Kirchhoff) stress resultants S_ϕ and S_θ, the angle of rotation Φ and the radial and axial displacements u and w are related to y as follows

$$\begin{aligned} &S_\phi / Ed = k^2 D(x, y)/4 , \qquad S_\theta / Ed = k^2 (xy' + y)/4 , \\ &\cos \Phi = y / D(x, y) , \qquad u = ak^2 x [xy' + y - \nu D(x, y)]/4 , \\ &w = a \int_x^1 \left[1 + \frac{1}{4} k^2 D(t, y(t)) - \nu \frac{\mathrm{d}}{\mathrm{d}t} (ty) \right] \sin \Phi(t) \, \mathrm{d}t . \end{aligned} \tag{2.17}$$

In contrast to the Föppl theory, the boundary value problem (2.16) contains, besides $Q(x)$, the load parameter $k = (2p_0 a / Ed)^{1/3}$. It is seen that in the limit case $k \to 0$ equations (2.16) reduce to Problems S and H for the Föppl theory (2.1), (2.3). For finite k, the solution structure of (2.16) turns out to be *markedly* different from that of (2.1), (2.3), in particular with respect to Problem H.

A mathematical analysis of Problems S and H defined by (2.16) was virtually nonexistent until 1980. Without surface load, the problem simplifies considerably. Two cases were treated by Clark and Narayanaswamy [8]: a membrane of revolution with a uniform radial edge load and a membrane with uniform edge load parallel to the axis of revolution. The first problem is reducible to a linear equation solvable in closed form. The second problem is nonlinear, it includes the special case of a flat annular membrane with transverse edge load treated by Schwerin [39] for the Föppl small deflection theory. In the presence of surface load, the integral equation technique used for the Föppl membrane equations was extended by Weinitschke [48] to obtain first existence results for $\nu = 0$ and a rather restricted range of boundary data.

A theory of existence and uniqueness of positive solutions for the boundary value problems (2.16) for the full range of physically meaningful data is still lacking. It has been argued that terms involving ν in the differential equations for finite rotations can often be neglected. Simmonds has given a rigorous justification of neglecting terms multiplied by ν in linear shell theory [40]. Although in the nonlinear theory his technique is not applicable, one might argue heuristically in the present case that the ν-term in the differential equation (2.16), which is $O(k^2)$, is small compared to the term $f(x, y)$, as the dominant load component acts in the direction normal to the membrane. For sufficiently small k this is certainly justified, but the influence of that term might increase for larger values of k. In the remaining part of this paper, we omit the term $2\nu k^2 \bar{p} Q/D$ and refer to (2.16) as the *simplified* Reissner theory.

The nonlinear function $f(x, y)$ has the same monotonicity property as in the Föppl theory because of

$$\frac{\partial f}{\partial y} = -\frac{k^2x^2Q^2(x)}{[y^2 + k^2x^2Q^2(x)]^{3/2}} \le 0 . \tag{2.18}$$

Hence the integral equation method employed for the Föppl theory is applicable so long as the boundary conditions are linear, that is for Problem S, $S > 0$ and for Problem H, $H > 0$, $\nu = 0$. The integral equation equivalent to $Ly = f$ and boundary conditions is

$$y(x) = q + \int_0^1 K(x, t)\tilde{f}(t, y(t))\,\mathrm{d}t + q_0[y^2(1) + \Theta^2]^{1/2} =: Ty , \tag{2.19}$$

where $0 \le x \le 1$, $\tilde{f}(t, y) := [1 - y/D(t, y)]/t$ and $\Theta := k|Q(1)|$. The Green's function $K(x, t)$ is defined by

$$K(x, t) = \begin{cases} 2k^{-2}(1 + nx^2)(t/x)^2 , & 0 \le t \le x \le 1 , \\ 2k^{-2}(1 + nt^2) , & 0 \le x \le t \le 1 . \end{cases} \tag{2.20}$$

For Problem S, we have

$$q = S , \qquad q_0 = 0 , \qquad n = -1 . \tag{2.21}$$

For Problem H, we have

$$q = H , \qquad q_0 = \nu , \qquad n = +1 . \tag{2.22}$$

$K(x, t)$ is essentially the same as $G(x, t)$ defined in (2.8), with $\nu = 0$ in Problem H.

Problem S and Problem H for $\nu = 0$ can now be solved as before. In view of $K \geq 0$ and (2.18) the operator T defined in (2.19) is antitone, so that Lemma 2.3 and Lemma 2.4 hold, provided $q > 0$. Application of the Banach fixed point theorem yields a theorem analogous to Theorem 2.5. The integral equation (2.19) has a unique positive solution for restricted ranges $S > S_0(k)c^{1/3}$, Problem S, and $H > H_0(k)c^{1/3}$, Problem H. In these ranges of S and H, the sequence $y_n(x)$ defined by $y_{n+1} = Ty_n$ converges uniformly to the solution $y(x)$, and $y \in M_n$ for $n = 0, 1, \ldots$ [49].

The restriction on S and H can again be removed by employing the Schauder fixed point theorem. The arguments are exactly as in the Föppl theory, the uniqueness follows as in (2.12), (2.13), replacing M by

$$M(x) = \frac{4(y_1 + y_2)Q^2(x)}{D(x, y_1)D(x, y_2)(y_1 D(x, y_2) + y_2 D(x, y_1))} \geq 0 . \tag{2.23}$$

THEOREM 2.10. *In the simplified Reissner theory of finite rotations, Problem S, or equivalently the integral equation* (2.19) *has a unique solution* $y(x)$ *for all* $S > 0$. *The solution is positive and* $y \in M_n$, $n = 0, 1, 2, \ldots$ [49].

For Problem H, this theorem only holds for $\nu = 0$, $H > 0$. The interpolated iteration (2.15) also applies to the simplified Reissner theory, rigorously for Problem S, that is, Theorem 2.9 remains valid. We have also tested (2.15) numerically in Problem H for $\nu \neq 0$ and found convergence for $H \geq 0$.

In the general case of Problem H, $\nu \neq 0$, equation (2.19) is a nonstandard integral equation because of the algebraic term multiplying $q_0 = \nu$. A complete solution of Problem H is due to Grabmüller and Pirner [19] and to Beck [3]. The uniqueness of positive solutions is obtained as follows. After multiplying by $x^3 w$, we integrate (2.12) over $(0, 1)$ and integrate by parts to get for $w = y_1 - y_2$

$$w'(1)w(1) = \int_0^1 x^3[w'^2(x) + M(x)w^2(x)]\,dx , \tag{2.24}$$

where $M(x)$ is defined by (2.23). Unless $w \equiv 0$, the right-hand side is strictly positive. In Problem S, $w(1) = 0$ and uniqueness follows. In Problem H, we have

$$w'(1) + w(1) - \nu[D(1, y_1(1)) - D(1, y_2(1))] = 0 . \tag{2.25}$$

Applying the mean-value theorem, we obtain $w'(1) + (1 - \nu\kappa)w(1) = 0$ where

$$0 < \kappa := \frac{y_\theta(1)}{[y_\theta^2(1) + k^2 Q^2(1)]^{1/2}} \leq 1 , \tag{2.26}$$

$y_\theta = y_1 + \theta(y_2 - y_1)$ with an intermediate variable $\theta(x)$. It is seen that y_θ is bounded from below by $\min\{\min y_1(x), \min y_2(x)\} > 0$. This yields $w'(1)w(1) = -(1 - \nu\kappa)w^2(1) \leq 0$ since $0 \leq \nu \leq 1/2$, and hence $w \equiv 0$ by equation (2.24).

We proceed to outline the existence proof. To this end, an extended version of a theorem concerning positive solutions of integral equations due to Novak [18] is utilized. Letting $g(x) := y(x) - q$, the objective is to determine solutions g of

$$g(x) = \int_0^1 K(x,t)\tilde{f}(t, g(t)+q)\,\mathrm{d}t + q_0[(g(1)+q)^2 + \Theta^2]^{1/2}, \tag{2.27}$$

satisfying $g(x)+q \geq 0$. Existence of solutions of (2.27) are obtained via Schauder's fixed point theorem. Consider the convex set M_δ defined in the Banach space $C[0,1]$

$$\begin{aligned} M_\delta = \Big\{ & g \in C[0,1] \mid \exists H \in C[0,1] \text{ with } 0 \leq H(t) \leq k/\delta \text{ and} \\ & g(x) = \int_0^1 K(x,t)H(t)\,\mathrm{d}t + q_0[\rho^2 + \Theta^2]^{1/2},\ 0 \leq \rho \leq \rho_0 \Big\}. \end{aligned} \tag{2.28}$$

Here $\delta > 0$, to be chosen later, and $\rho_0 > 0$ is defined by

$$\rho_0 = \frac{K_0 + |q| + \Theta}{1 - q_0}, \qquad K_0 := \frac{k}{\delta} \max_{0 \leq x \leq 1} \int_0^1 K(x,t)\,\mathrm{d}t \leq \frac{8}{3\delta k}. \tag{2.29}$$

One can show that, if g is a solution of (2.27) satisfying $g(x)+q \geq \delta > 0$ then $|g(x)| \leq \rho_0$ and $g \in M_\delta$. Hence the solution set of (2.27) is a subset of M_δ. Next one introduces an operator W acting on M_δ by

$$(Wg)(x) := \int_0^1 K(x,t)\tilde{f}(t, d_g(t))\,\mathrm{d}t + q_0[(g(1)+q)^2 + \Theta^2]^{1/2}. \tag{2.30}$$

A similar operator and the cutoff function $d_g(x) := \max\{g(x)+q, \delta\}$, $0 \leq x \leq 1$ had first been introduced by Novak [18] within the context of annular Föppl membrane problems. It is not difficult to prove the following

THEOREM 2.11 [18]. *Suppose there exists a constant $\delta > 0$ such that for any $g \in M_\delta$ satisfying* $\min_{0 \leq x \leq 1}[g(x)+q] \leq \delta$ *we have the property*

$$\exists x_0 \in [0,1]\colon\ g(x_0) < (Wg)(x_0). \tag{N}$$

Then equation (2.19) *has a solution $y(x) > \delta$.*

A sufficient condition for the existence of positive solutions of Problem H is now obtained by choosing H such that condition (N) is satisfied.

THEOREM 2.12 [18]. *Assume that either $H \geq 0$ or*

$$H < 0 \quad \text{and} \quad \int_0^1 K(1,t)\tilde{f}\Big(t, -H\,\frac{1+t^2}{2t^2}\Big)\,\mathrm{d}t + \nu\Theta > -H, \tag{2.31}$$

then there exists a number $\delta > 0$ such that condition (N) *holds for Problem H.*

In the Föppl theory, solutions of Problem H exist for all real H (Theorem 2.8). This result does not carry over to the simplified Reissner theory. Indeed, a tensile solution y of Problem H necessarily satisfies $y(x) \geq y(1) > 0$. But from (2.19) one has, if $Q^2(x) > 0$,

$$0 < (1-\nu)y(1) \leq H + \int_0^1 K(1,t)\,\frac{\mathrm{d}t}{t} + \nu\Theta, \tag{2.32}$$

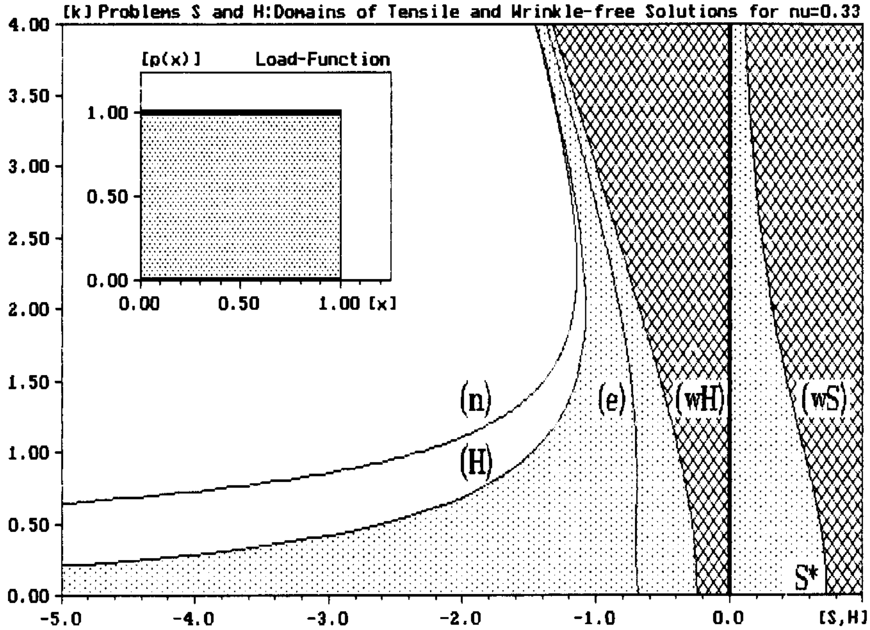

Fig. 1. The arcs (n) and (e) denote, respectively, the curves Γ_n and Γ_e which are approximations for the *separatrix* (H) between the domains of existence (dotted) and nonexistence (blank) of tensile solutions to Problem H. The domains of wrinkle-free solutions (crosshatched) which extend unboundedly to the right are bounded to the left by the arcs (wS) for Problem S and (wH) for Problem H. The load is uniform and Poisson's number is $\nu = 1/3$.

which provides a necessary condition for the existence of a solution to Problem H. Unfortunately the integral in (2.31) cannot be expressed in closed form, even in the case of uniform pressure $Q = 1$. But the important point is that (2.31) and (2.32) yield graphs Γ_e and Γ_n in the (H, k) plane bounding the domains of *existence* and *nonexistence* of tensile solutions of Problem H, respectively. For $Q = 1$, these graphs are shown in Fig. 1 (for $\nu = 1/3$).

It is clear from the above that there must be an arc Γ in the (H, k) plane, located between Γ_e and Γ_n in Fig. 1, which separates precisely the domains of existence and nonexistence. Indeed, since the solutions of Problem H are contained in those of Problem S for $S > 0$, we simply need to solve Problem S for $S = 0$ and determine the corresponding value of $H = H(k)$ from (2.16). As an analytic solution of (2.19) is not available, we solve the problem numerically. This was done by Weinitschke in [45], although an existence proof for Problem S, $S = 0$ was still lacking. Recently, Beck [3] has supplied such a proof, which is by no means simple. The separation curve Γ, whose existence and geometric properties were also proved in [3], is included in Fig. 1, again for the case $Q = 1$, $\nu = 1/3$.

3. Annular membranes

In this section we consider annular membranes of inner radius b and outer radius a under vertical pressure $p = p(r)$. At the edges, either the radial stress or the radial displacement is prescribed. The governing equations (1.6) and (1.8) are the same, and so are the dimensionless variables, except that the interval $(0, 1)$ is replaced by $(\varepsilon, 1)$ where $\varepsilon = b/a$. This also changes the lower limit in the integral for Q. Thus the annular membrane problems can be reduced to the differential equation

$$Ly = f(x, y) := \begin{cases} (2/y^2)R^2(x, \varepsilon): & \text{Föppl theory}\,, \\ (2/kx)^2[1 - y/D(x, y)]: & \text{simplified Reissner theory}\,, \end{cases}$$

$$0 < \varepsilon < x < 1\,, \tag{3.1}$$

$$R(x, \varepsilon) := \frac{2}{x^2}\int_\varepsilon^x t\bar{p}(t)\,\mathrm{d}t\,, \qquad D(x, y) := [y^2 + k^2x^2R^2(x, \varepsilon)]^{1/2}\,.$$

The radial and circumferential stress components σ_r and σ_θ, and the radial and normal displacements u and w are related to y as in (2.2) and (2.17), provided $Q(x)$ is replaced by $R(x, \varepsilon)$ in all the formulas. If the surface load is *uniform*, then $R(x, \varepsilon) = 1 - \varepsilon^2/x^2$. A pair of boundary conditions at $x = \varepsilon$ and $x = 1$ is taken, respectively, from

$$y(\varepsilon) = s \quad \text{or} \quad \varepsilon y'(\varepsilon) + (1 - \nu)y(\varepsilon) = h\,, \tag{3.2}$$

at the inner edge and

$$y(1) = S \quad \text{or} \quad \begin{cases} y'(1) + (1 - \nu)y(1) = H: & \text{Föppl}\,, \\ y'(1) + y(1) - \nu D(1, y(1)) = H: & \text{simplified Reissner}\,, \end{cases} \tag{3.3}$$

at the outer edge. Introducing the notations (s, S), (s, H), (h, S) and (h, H) reference shall be made in an obvious manner to the four different boundary value problems arising from (3.1), (3.2) and (3.3).

The analysis of annular membrane problems defined by (3.1) and (3.2) for $s \geq 0$, $S \geq 0$, h and H real, and for an arbitrary load $\bar{p}(x)$ is more complicated than in the circular membrane problems since solutions $y(x)$ are permitted to become zero at the edges of the annulus. The *regular tensile solutions* of (3.1) (rt-solutions for short, which have to be understood as functions $y \in C^2(\varepsilon, 1) \cap C^1[\varepsilon, 1]$ satisfying $y(x) > 0$ for $\varepsilon \leq x \leq 1$) now have to be distinguished from *regular nonnegative solutions* (rn-solutions) which suffer a loss of regularity $y \in C^2(\varepsilon, 1) \cap C^0[\varepsilon, 1]$ and satisfy $y(x) > 0$ only for $\varepsilon < x < 1$. Furthermore, the solutions $y(x)$ are not necessarily monotone in the interval $(\varepsilon, 1)$, there is no counterpart of inequality (2.4) here. The four different boundary parameters s, h, S and H considerably enhance the variety of solution behavior.

As in Section 2, we begin by discussing results for the Föppl *small finite deflection theory*. The first solution of (3.1)–(3.3) for uniform pressure and $s = H = 0$ was given by Schwerin [39] in terms of a formal power series. In the absence of surface loads, he also found a closed form solution for a membrane subjected to axial edge load and fixed edges ($h = H = 0$). To simplify his calculations, he transformed the differential equation (3.1) into the form $U''(\xi) = -\xi^2/U^2$. This transformation turned out to be crucial for the theoretical analysis of rn-solutions for both Föppl and Reissner theory. We write the *Schwerin transformation* for our purposes as follows

$$z = \frac{x^2 - \varepsilon^2}{1 - \varepsilon^2}\,, \qquad g(z) = \omega^4x^2y(x)\,, \quad \varepsilon \leq x \leq 1\,, \quad \omega := (1 - \varepsilon^2)^{-1/3}\,. \tag{3.4}$$

This change of variables maps any rn-solution $y(x)$ of (3.1) into an rn-solution $g(z)$ of the differential equation

$$-\frac{d^2g}{dz^2} = F(z, g) := \begin{cases} 2P^2(z,\varepsilon)/g^2: & \text{Föppl}\,, \\ RS(z, g): & \text{simplified Reissner}\,, \end{cases} \quad 0 < z < 1\,, \tag{3.5}$$

where the following notations are used:

$$P(z, \varepsilon) := \frac{1}{1-\varepsilon^2} \int_\varepsilon^{x(z)} t\bar{p}(t)\,dt\,, \qquad x(z) := [z(1-\varepsilon^2) + \varepsilon^2]^{1/2}\,,$$

$$RS(z, g) := \left(\frac{1}{k\omega x(z)}\right)^2 \left[1 - \frac{g}{\sqrt{g^2 + \Theta^2(z)}}\right], \qquad \Theta(z) := 2k\omega x(z) P(z, \varepsilon)\,.$$

The boundary conditions (3.2), (3.3) can be expressed in terms of z and $g(z)$ by

$$g(0) = s \quad \text{or} \quad B_0[g] = h\,, \qquad g(1) = S \quad \text{or} \quad B_1[g] = H\,. \tag{3.6}$$

Here the new boundary parameters $s \geqslant 0$, $S \geqslant 0$, h and H real, are obtained by suitably stretching the old ones. The boundary operators $B_j[g]$ have the form

$$B_0[g] := \varepsilon^2 \dot{g}(0) - \mu g(0)\,, \qquad \mu := \tfrac{1}{2}(1-\varepsilon^2)(1-\nu)\,,$$

$$B_1[g] := \begin{cases} \dot{g}(1) - \mu g(1): & \text{Föppl}\,, \\ \dot{g}(1) - \dfrac{1-\varepsilon^2}{2}\,[g(1) + \nu\sqrt{g^2(1) + \Theta^2(1)}]: & \text{simplified Reissner}\,. \end{cases}$$

Unless otherwise stated, a dot denotes differentiation with respect to z. Since $F(z, g)$ is nonnegative it is clear from (3.5) that any rn-solution $g(z)$ must be concave. Thus, $g(z)$ with $g(0) = s$ and $g(1) = S$ is supported by the linear function $q(z) := zS + (1-z)s$ in the sense that $g(z) - q(z) \geqslant 0$ holds.

The convergence of Schwerin's series solution of $\ddot{g} = -z^2/2g^2$, ($\bar{p} = 1$), was proved by Weinitschke in [48], but only for the case $s = 0$ and S or H sufficiently large, not covering the situation $H = 0$. An integral equation method analogous to (2.7) was also employed in [48] for the solution of Problem (s, H). However, the results obtained by applying the Banach fixed point theorem will not be discussed here, as they impose unacceptable restrictions on the parameters ε, ν, s and H, excluding, in particular, the case $s = H = 0$. As for circular membranes, a constructive existence proof for an rt-solution (not covering a stress-free inner edge $s = 0$) as a limit of $y_n(x)$ defined by (2.15), with an appropriate definition of T, has been given by Novak [33].

The solutions for a free inner edge $s = 0$ and uniform load $\bar{p} = 1$ can be expressed in terms of solutions of the circular membrane, as shown by Grabmüller and Weinitschke [21]. More precisely, we have the following

THEOREM 3.1. *Let $z(\xi)$ be the solution of* (2.1), (2.3), *Problem S with $z(1) = S_\varepsilon := (1-\varepsilon^2)^{-4/3} S > 0$, $Q = 1$. Then there exists a unique solution $y(x)$ of Problem* (s, S), $s = 0$, $\bar{p} = 1$. *This solution is positive and has the form*

$$y(x) = (1-\varepsilon^2)^{1/3} \left(1 - \frac{\varepsilon^2}{x^2}\right) z(\xi)\,, \qquad \xi = \left(\frac{x^2 - \varepsilon^2}{1-\varepsilon^2}\right)^{1/2}\,. \tag{3.7}$$

Let $z(\xi)$ *be the solution of* (2.1), (2.3), *Problem H with* $z'(1)+(1-\nu_\varepsilon)z(1)=H_\varepsilon := (1-\varepsilon^2)^{-1/3}H$, *where* $\nu_\varepsilon := \nu - \varepsilon^2(1+\nu)$, $Q=1$. *Then there exists a unique solution of Problem* (s, H), $s=0$, $\bar{p}=1$. *This solution is positive and has the form* (3.7).

In particular, this theorem covers the physically interesting cases $s=0$, $S>0$ and $s=0$, $H\geqslant 0$, but only for uniform load. It also proves a formal result obtained by Schwerin [39] concerning the stress concentration factor. In addition, existence and uniqueness of rt-solutions for Problem (s, S), $s>0$, $S>0$ and for Problem (s, H) for $s>0$ and $s+H>0$, in terms of the new parameters defined in (3.6), for arbitrary load functions $\bar{p}(x)$, was proved in [21]. This was done by reducing these two problems to integral equations with antitone operators and applying Schauders fixed point theorem, along the lines described in Section 2.

Concerning the more difficult Problems (h, S) and (h, H), a new device was brought into the analysis of rt-solutions by E. Novak [33] who considered positive solutions of certain nonlinear integral equations. His method led to a considerable improvement of the known existence results. The standard method of using Green's function shows that rt-solutions $g(z)$, for example of Problem (s, S), coincide with the positive $C^0[0,1]$-solutions of the integral equation

$$g(z)=q(z)+\int_0^1 k(z,t)F[t,g(t)]\,\mathrm{d}t\,,\quad 0\leqslant z\leqslant 1\,, \tag{3.8}$$

where $q(z)$ is defined above and where

$$k(z,t)=\begin{cases} t(1-z): & 0\leqslant t\leqslant z\leqslant 1\,,\\ z(1-t): & 0\leqslant z\leqslant t\leqslant 1\,.\end{cases}$$

A new dependent variable $f(z) := g(z)-q(z)$ is introduced, and instead of (3.8) the following integral equation is considered (see (2.30)):

$$f(z)=(Wf)(z):=\int_0^1 k(z,t)F[t,d_f(t)]\,\mathrm{d}t\,,\quad 0\leqslant z\leqslant 1\,. \tag{3.9}$$

Here, for some $\delta>0$, the cutoff function $d_f(z):=\max\{f(z)+q(z),\delta\}$, $0\leqslant z\leqslant 1$, provides a lower bound for the solutions of (3.9). As in Section 2 a convex subset $M\subset C^0[0,1]$ is considered which consists of all functions f admitting an integral representation $f(z)=\int_0^1 k(z,t)H(t)\,\mathrm{d}t$ for some $H\in C^0[0,1]$, $H\geqslant 0$. An application of the Schauder fixed point theorem then yields an analogon to Theorem 2.11, with $g(x)$ replaced by $f(z)$ and W defined by (3.9), proving that equation (3.9) has a $C^0[0,1]$-solution $f(z)$ with $f+q>\delta$.

It is clear that Novak's condition (N) must be void if $s>0$ and $S>0$ is considered. Thus, a new *existence* proof for rt-solutions to Problem (s, S) is obtained because it is seen from (3.9) that $g(z):=f(z)+q(z)$ solves the integral equation (3.8). In [17], the question of *uniqueness* of rt-solutions to each boundary value problem was fully resolved by a suitable application of Hopf's generalized maximum principle [36]. The above new integral equation method was also extended to Problems (s, H), (h, S) and (h, H). The particular constellation of Problem (s, H) again leads to a void condition (N). In summary, we have [17, 18]:

THEOREM 3.2. *Let* $P^2(1,\varepsilon)>0$, *then in the small finite deflection theory both Problem* (s, S) *and Problem* (s, H) *have a unique rt-solution for all* $s>0$, $S>0$ *and* H *real. Moreover, rt-solutions for Problems* (h, S) *and* (h, H) *are unique.*

Analogous existence results do not hold in the cases of Problems (h, S) and (h, H) for the Föppl theory. A discussion of the crucial condition (N) surprisingly shows that rt-solutions are absent in an unbounded, simply connected subset of the respective parameter ranges (h, S) and (h, H). This was first discovered by Grabmüller and Novak [18], who estimated the domains of existence and nonexistence of rt-solutions by simple analytical curves. Unfortunately, a gap was left in the parameter ranges where a definite statement on existence or nonexistence was impossible. We shall return to this point after having commented on Reissner's finite rotation theory for annular membrane problems.

Before 1980, a mathematical analysis of the boundary value problems posed by the simplified Reissner theory seemed to be nonexistent as far as annular membranes under a nonvanishing surface load are concerned. First existence results for a rather restricted range of the boundary data were obtained by Weinitschke [48], who extended the integral equation technique formerly used for the Föppl membrane equations. The discovery of Novak's device led to a marked improvement and extension of these results. The general analysis developed above covers also the membrane equations of Reissner's theory and thus provides the existence of rt-solutions to Problem (s, S) for each $s > 0$, $S > 0$. This result was established by Grabmüller and Pirner [19]. The uniqueness of rt-solutions was also proved in [19] for the whole set of physically meaningful boundary parameters $s > 0$, $S > 0$, h and H real, and for each of the problems defined by (3.1) and (3.3). In summary, there is the following

THEOREM 3.3. (a) *Let $P^2(1, \varepsilon) > 0$, then Problem (s, S) has a unique rt-solution for all $s > 0$ and $S > 0$.*
(b) *Problems (s, H), (h, S) and (h, H) have at most one rt-solution for all $s > 0$, $S > 0$, h and H real.*

Problems (s, H) and (h, H) differ from those of the Föppl membrane model since the boundary operator $B_1[g]$ in (3.6) now is genuinely nonlinear. The integral equation (3.8) changes to the form

$$g(z) = q_j(z) + \int_0^1 k_j(z, t) RS[t, g(t)]\, dt + w_j(z)\sqrt{g^2(1) + \Theta^2(1)}\,, \quad 0 \leq z \leq 1\,, \tag{3.10}$$

with appropriate Green's functions $k_j(z, t)$ and with linear functions $q_j(z) := A_j z + B_j$, $w_j(z) := \nu(C_j z + D_j)$, dependent on the boundary parameters s, S, h and H. The subscripts $j = 1, 2, 3$ refer to Problems (s, H), (h, S) and (h, H), respectively. The operator W in (3.9) needs to be corrected by an additional term $w_j(z)\{[f(1) + q_j(1)]^2 + \Theta^2(1)\}^{1/2}$ and the subset $M \subset C^0[0, 1]$ must be suitably modified in order to show that Theorem 2.11 now holds for annular membranes within Reissner's theory. Details are elaborated in [19], and the results obtained from a discussion of condition (N) differ from those of the Föppl membrane model rather quantitatively than qualitatively. As before, the ranges of boundary parameters associated with Problems (h, S) and (h, H) are subdivided into the three simply connected subsets of existence, of nonexistence and a remainder of unknown relation to rt-solutions. Surprisingly, the behavior of Problem (s, H) alters significantly when the theory turns over from small finite deflections to finite rotations. Contrary to the statement of Theorem 3.2, the parameter range $s > 0$, H real, now splits in the same manner as described above. An explanation stems from the concept of superfunctions which was more thoroughly discussed in [14].

A concave function $g_\infty \in C^2(0,1) \cap C^0[0,1]$ is said to be a *superfunction* of Problem (s, S) if for any rn-solution $g(z)$ the relations $g(j) = g_\infty(j)$ with $j = 0, 1$, and $g(z) \leq g_\infty(z)$ with $0 \leq z \leq 1$ hold. For $s \geq s_0$ and $S \geq S_0$, a superfunction $g_\infty(z)$ of Problem (s, S) is generated by $g_\infty(z) := (1-z)(s-s_0) + z(S-S_0) + g_{\infty,0}(z)$ where $g_{\infty,0}(z)$ denotes any superfunction of Problem (s_0, S_0). For example, suitable superfunctions of Problem (s = 0, S = 0) are provided by [14]

$$g_{\infty,0}(z) := \begin{cases} \sqrt{z(1-z)}: & \text{Föppl}\,, \\ -\dfrac{\omega^4}{k^2}\left[z \ln \varepsilon^2 + 2x^2(z) \ln\left(\dfrac{x(z)}{\varepsilon}\right)\right]: & \text{simplified Reissner}\,. \end{cases} \tag{3.11}$$

Pursuing an idea mentioned in Section 2, rt-solutions $g = g(z; s, S)$ of Problem (s, S) may be interpreted as rt-solutions of Problem (s, H) for a value H given by $B_1[g(\cdot\,; s, S)] = H$. The superfunction $g_\infty(z) := q(z) + g_{\infty,0}(z)$ with $g_{\infty,0}$ taken from (3.11) is appropriate to verify $q(z) \leq g(z; s, S) \leq g_\infty(z)$ and to show $B_1[g(\cdot\,; s, S)] \to +\infty$ as $S \to +\infty$. In addition, taking account of

$$\dot g_\infty(1) \leq \dot g(1; s, S) \leq \dot q(1) - \limsup_{z \to 0+} \frac{1}{z} \int_0^z k(1-z, t) F[t, g_\infty(t)]\, dt\,, \tag{3.12}$$

an elementary calculation yields $\lim_{S\to 0+} B_1[g(\cdot\,; s, S)] = -\infty$ for Föppl's theory, while

$$\lim_{S \to 0+} B_1[g(\cdot\,; s, S)] \geq \dot g_{\infty,0}(1) - s - \tfrac{1}{2}\nu(1-\varepsilon^2)|\Theta(1)| =: H_l > -\infty$$

in case of Reissner's theory [16]. In this case the range of the mapping $(s, S) \mapsto B_1[g(\cdot\,; s, S)]$ does not cover the whole real axis. So rt-solutions must be absent at least for $H < H_l$.

First ideas of the mapping argument used above originate from Pirner's diploma thesis [35] where only the finite rotation case was treated. However, the ingenious mapping idea was appropriate to fill in the gap mentioned above between the domains of existence and nonexistence of rt-solution. The following strategy was successful.

Firstly, the natural domain $Ex(s, S) := \{(s, S): s \geq 0, S \geq 0\}$ of existence of rn-solutions to Problem (s, S) is introduced. Theorem 3.3 confirms that the unique *rt-solutions* $g := g(z; s, S)$ of Problem (s, S) are defined at every interior point (s, S) of $Ex(s, S)$. At every boundary point of $Ex(s, S)$ there exists a unique *rn-solution*. This was shown by Pirner [35] in the finite rotation case and by Grabmüller [14] within a more general setting which also covers the small finite deflection case, see also [20].

THEOREM 3.4. *Assume $P^2(1, \varepsilon) > 0$, and let (s, S) be any boundary point of the set $Ex(s, S)$.*

(a) The small finite deflection model: *There exists a unique rn-solution $g = g(z; s, S)$ of Problem (s, S) with a finite derivative $\dot g(0; s, S)$ but an unbounded derivative $\dot g(1; s, S) = -\infty$ if $S = 0$.*
(b) The finite rotation model: *There exists a unique rn-solution $g = g(z; s, S)$ of Problem (s, S) with additional regularity $g \in C^1[0,1]$ for all $s \geq 0$, $S \geq 0$.*

Secondly, each boundary operator B_j, $j = 0, 1$, acting on the totality of rn-solutions $g(z; s, S)$ of Problem (s, S) is interpreted as a mapping $\beta_j(s, S) := B_j[g(\cdot\,; s, S)]$ from the domain $Ex(s, S)$ into the reals. If the images of the set $Ex(s, S)$ subject to the vectorial mappings $(s, \beta_1(s, S))$ and $(\beta_0(s, S), S)$ are, respectively, denoted by $Ex(s, H)$ and $Ex(h, S)$, it becomes evident that these sets can be expected to form the natural domains of existence of rn-solutions to Problems (s, H) and (h, S). The more involved Problem (h, H) will be discussed below.

Of major interest is the behavior of the semi-axes $s \geq 0$ and $S \geq 0$ under the mappings β_j since their images $\gamma_j(s) := \beta_j(s, 0)$ and $\Gamma_j(S) := \beta_j(0, S)$ should be expected to separate the domain of existence of rt-solutions from that of nonexistence.

The development of the necessary analysis initiated by Pirner [35] was concerned with the simplified Reissner model. Pirner's results were substantially based on the following theorem which summarizes the main properties of the mappings β_j.

THEOREM 3.5 [The finite rotation model]. *Let $s_0 \geq 0$ and $S_0 \geq 0$ be fixed.*

(a) *The projections $\beta_j(s_0, \cdot)$: $\bar{\mathbb{R}}_+ \to \mathbb{R}$ are strictly increasing and continuous, and so must be Γ_j. The ranges of $\beta_j(s_0, \cdot)$ are the segments $[\gamma_j(s_0), +\infty)$. Asymptotic forms of Γ_j are provided by*

$$\left.\begin{aligned} &0 \leq \Gamma_0(S) - \varepsilon^2 S = o(1) \\ &0 \leq (1-\mu)S - \Gamma_1(S) = o(1) \end{aligned}\right\} \quad \text{as } S \to +\infty\,. \tag{3.13}$$

(b) *The projections $\beta_j(\cdot\,, S_0)$: $\bar{\mathbb{R}}_+ \to \mathbb{R}$ are strictly decreasing and continuous, and so must be γ_j. The ranges of $\beta_j(\cdot\,, S_0)$ are the segments $(-\infty, \Gamma_j(S_0)]$. Asymptotic forms of γ_j are provided by*

$$\left.\begin{aligned} &0 \leq \gamma_0(s) + (\varepsilon^2 + \mu)s = o(1) \\ &0 \leq -(1-\varepsilon^2)\nu|\Theta(1)|/2 - s - \gamma_1(s) = o(1) \end{aligned}\right\} \quad \text{as } s \to +\infty\,. \tag{3.14}$$

A proof of the continuity and monotonicity statements in Theorem 3.5, which is by no means simple, was supplied in [20]. The asymptotic forms (3.13) and (3.14) are easily derived from the integral representation (3.8) using

$$\left.\begin{aligned} &\dot{g}(0; s, S) = S - s + \int_0^1 (1-t)RS[t, g(t)]\,dt\,, \\ &\dot{g}(1; s, S) = S - s - \int_0^1 tRS[t, g(t)]\,dt\,, \end{aligned}\right\} \tag{3.15}$$

and observing that $RS[z, q(z)] \to 0$ as $s \to +\infty$ or $S \to +\infty$. The last relation holds pointwise for each $z \in (0, 1)$ and for $q(z) := zS + (1-z)s$ supporting $g(z)$.

By Theorem 3.5, an rn-solution $g(z; s, S)$ corresponding to $(s, S) \in Ex(s, S)$ induces the displacements $h := \beta_0(s, S) \leq \Gamma_0(S)$ at the inner edge of the annulus, and $H := \beta_1(s, S) \geq \gamma_1(s)$ at the outer edge. Since the mappings $\beta_j(s, \cdot)$ and $\beta_j(\cdot\,, S)$ are one-to-one, the representations

$$Ex(s, H) = \{(s, H) \colon H \geq \gamma_1(s),\, s \geq 0\}\,, \qquad Ex(h, S) = \{(h, S) \colon h \leq \Gamma_0(S),\, S \geq 0\}$$

become evident which in turn allow to be interpreted as an existence result for both Problems (s, H) and (h, S).

THEOREM 3.6 [The existence statement of Problems (s, H) and (h, S) for the finite rotation theory]. *Assume* $P^2(1, \varepsilon) > 0$. *Let, respectively,* $g_0(z; S)$ *and* $g_1(z; s)$ *denote the unique rn-solutions of Problems* $(s = 0, S)$ *and* $(s, S = 0)$, *and define the continuous curves*

$$\gamma_1(s) := \dot{g}_1(1; s) - \frac{1-\varepsilon^2}{2}\,\nu|\Theta(1)|\,, \quad s \geq 0\,, \qquad \Gamma_0(S) := \varepsilon^2 \dot{g}_0(0; S)\,, \quad S \geq 0\,. \tag{3.16}$$

Then the parameter ranges of Problems (s, H) *and* (h, S) *are decomposed, respectively, by* γ_1 *and* Γ_0 *into complementary subsets of existence and nonexistence as follows*:

(a) *Problem* (s, H) $((h, S)$, *resp.) has a unique rt-solution* $g(z)$ *if and only if* $s > 0$ *and* $H > \gamma_1(s)$ $(S > 0$ *and* $h < \Gamma_0(S)$, *resp.).*

(b) *At any boundary point* $(s \geq 0, H = \gamma_1(s))$ *a unique rn-solution* $g(z)$ *of Problem* (s, H) *is given by* $g_1(z; s)$ *and thus satisfies* $g(0) = s$ *and* $g(1) = 0$. *At any boundary point* $(s = 0, H)$, $H > H_0 := \gamma_1(0)$, *there exists a unique rn-solution* $g(z)$ *which satisfies* $g(0) = 0$ *and* $g(1) > 0$.

(c) *At any boundary point* $(h = \Gamma_0(S), S \geq 0)$ *a unique rn-solution* $g(z)$ *of Problem* (h, S) *is given by* $g_0(z; S)$ *and thus satisfies* $g(0) = 0$ *and* $g(1) = S$. *At any boundary point* $(h, S = 0)$, $h < h_0 := \Gamma_0(0)$, *there exists a unique rn-solution* $g(z)$ *which satisfies* $g(0) > 0$ *and* $g(1) = 0$.

(d) *Outside the sets* $Ex(s, H)$ *and* $Ex(h, S)$ *rn-solutions cannot exist.*

It is worth noting that the relations (3.13) and (3.14) supplemented by the superfunction (3.11) provide lower and upper bounds for the *separatrices* $\gamma_1(s)$ and $\Gamma_0(S)$. Indeed, letting $\gamma_1^*(s) := -s - (1 - \varepsilon^2)\nu|\Theta(1)|/2$, a simple calculation yields

$$\left.\begin{aligned} &\gamma_1^*(s) + \dot{g}_{\infty,0}(1) \leq \gamma_1(s) \leq \gamma_1^*(s)\,, \quad s \geq 0\,,\\ &\varepsilon^2 S \leq \Gamma_0(S) \leq \varepsilon^2(S + \dot{g}_{\infty,0}(0))\,, \quad S \geq 0\,. \end{aligned}\right\} \tag{3.17}$$

The bounds (3.17) are simpler, but slightly coarser than those given in the paper [19]. The domains of existence of tensile solutions of Problem (h, S) for various k are illustrated in Fig. 2.

The analysis of Problem (h, H) is more complex because the images of the semi-axes $s \geq 0$ and $S \geq 0$ in (h, H)-plane now are *parametrized* arcs

$$\Sigma' := \{(h, H) \colon h = \gamma_0(s),\ H = \gamma_1(s),\ s \geq 0\}\,,$$

$$\Sigma'' := \{(h, H) \colon h = \Gamma_0(S),\ H = \Gamma_1(S),\ S \geq 0\}\,,$$

which form a connected continuous arc $\Sigma := \Sigma' \cup \Sigma'' \subset (h, H)$. To make the representation of Σ *explicit*, Theorem 3.5 is utilized. The functions

$$\rho_1(h) := \gamma_1[\gamma_0^{-1}(h)]\,, \quad h \leq h_0 \quad \text{and} \quad \rho_2(h) := \Gamma_1[\Gamma_0^{-1}(h)]\,, \quad h \geq h_0\,,$$

are properly defined and map onto the segments $(-\infty, H_0]$ and $[H_0, +\infty)$, respectively. This shows the (h, H)-plane is complementarily subdivided by the arc

$$\Sigma = \{(h, H) \colon H = \rho_1(h) \text{ for } h \leq h_0 \text{ and } H = \rho_2(h) \text{ for } h > h_0\}$$

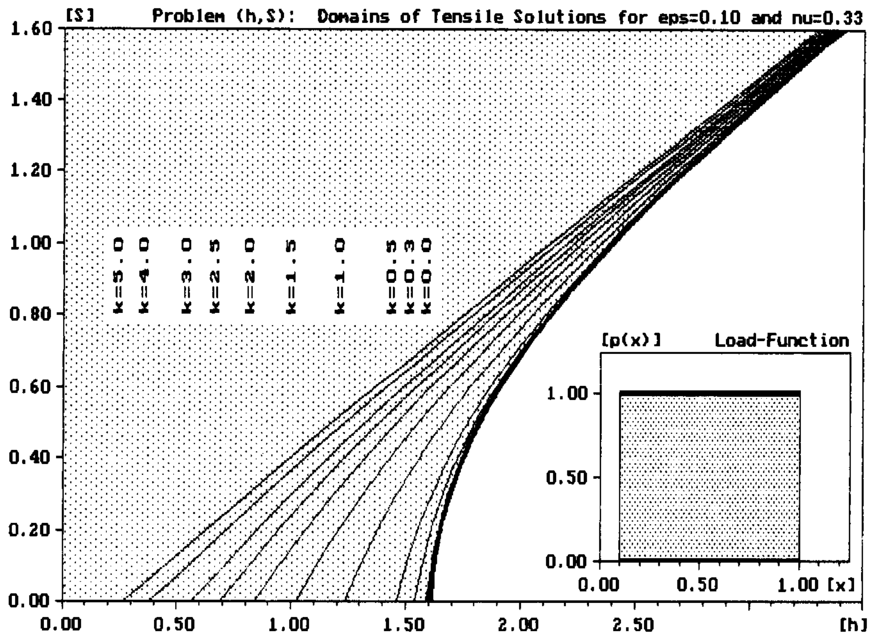

Fig. 2. Tensile solutions $y(x)$ of Problem (h, S) only exist for boundary parameters h and S within the dotted domain which extends unboundedly to the left. The diagram shows how the *separatrix* $\Gamma_0(S)$ varies with k. For $k = 0$ (*Föppl model*) the domain of existence is maximal. The surface load is uniform, $\nu = 1/3$, and $\varepsilon = 0.1$.

into two subdomains, which clearly are the domains of existence and nonexistence of rt-solutions to Problem (h, H). Via an implicit-function argument the following set is recognized as the domain of *existence*:

$$Ex(h, H) := \{(h, H) \colon H \geq \rho_1(h) \text{ for } h \leq h_0,\ H \geq \rho_2(h) \text{ for } h > h_0\} . \tag{3.18}$$

Here, the details are omitted and reference is made to [20]. Again, the results can be interpreted as an existence theorem.

THEOREM 3.7 [The existence statement of Problem (h, H) for the finite rotation theory]. *In addition to the assumptions of Theorem* 3.6, *define*

$$\left.\begin{aligned} &\gamma_0(s) := \varepsilon^2 \dot{g}_1(0; s) - \mu s\,, \quad s \geq 0\,, \\ &\Gamma_1(S) := \dot{g}_0(1; S) - \frac{1-\varepsilon^2}{2}\,[S + \nu\sqrt{S^2 + \Theta^2(1)}]\,, \quad S \geq 0\,. \end{aligned}\right\} \tag{3.19}$$

Then the parameter range of Problem (h, H) is decomposed by Σ into complementary subsets of existence and nonexistence as follows:

(a) *A unique rt-solution $g(z)$ exists if and only if*

$$H > \begin{cases} \rho_1(h) := \gamma_1[\gamma_0^{-1}(h)]\,, & h \leq h_0 := \gamma_0(0)\,, \\ \rho_2(h) := \Gamma_1[\Gamma_0^{-1}(h)]\,, & h > h_0\,. \end{cases}$$

(b) *At any boundary point $(h \leq h_0,\ H = \rho_1(h))$ a unique rn-solution $g(z)$ is given by*

$g_1(z; s)$ *and thus satisfies* $g(0) = s = \gamma_0^{-1}(h) > 0$ *and* $g(1) = 0$. *At any boundary point* $(h > h_0,\ H = \rho_2(h))$, *a unique rn-solution* $g(z)$ *is given by* $g_0(z; S)$ *and thus satisfies* $g(0) = 0$ *and* $g(1) = S = \Gamma_0^{-1}(h) > 0$.

(c) *Outside the set* $Ex(h, H)$ *rn-solutions cannot exist.*

The domains of existence of tensile solutions are given in Fig. 3, for various values of k.

Lower and upper bounds for the separatrix Σ are derived from a fundamental relation between rn-solutions $g(z)$ and any superfunction $g_\infty(z)$ of Problem (s, S). Notice that $q(z) \leq g(z) \leq g_\infty(z)$ holds. Taking finite differences at $z = 0$ and $z = 1$, in the limit the derivatives are subjected to the following inequalities:

$$\dot{g}_\infty(0) \geq \dot{g}(0) \geq S - s \geq \dot{g}(1) \geq \dot{g}_\infty(1)\,. \tag{3.20}$$

Using this and the function $\gamma_1^*(s)$ defined above one obtains straightforwardly

$$\left.\begin{aligned} &\dot{g}_{\infty,0}(0) \geq \frac{1}{\varepsilon^2} B_0[g] + \left(1 + \frac{\mu}{\varepsilon^2}\right) s - S \geq 0\,, \\ &B_1[g] \geq (1-\mu)S + \dot{g}_{\infty,0}(1) + \gamma_1^*(s)\,. \end{aligned}\right\} \tag{3.21}$$

From these relations and from (3.17) the following bounds are obtained for the strictly increasing arc $\rho_1(h)$, where $\rho_1^*(h) := H/(\varepsilon^2 + \mu) - (1 - \varepsilon^2)\nu|\Theta(1)|/2$ will be used

$$\rho_1^*(h) + \dot{g}_{\infty,0}(1) - \frac{\varepsilon^2}{\varepsilon^2 + \mu}\, \dot{g}_{\infty,0}(0) \leq \rho_1(h) \leq \rho_1^*(h)\,, \quad h \leq h_0\,. \tag{3.22}$$

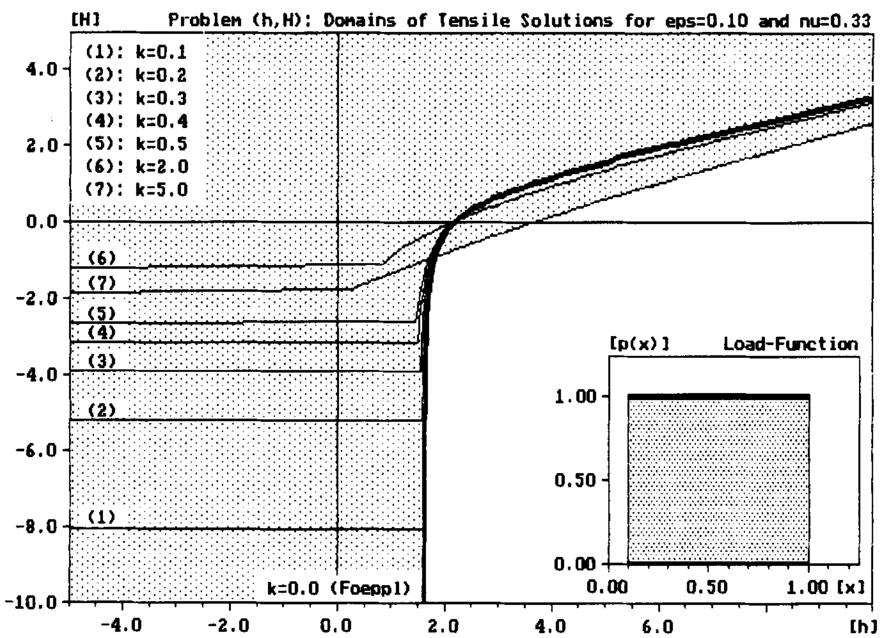

Fig. 3. Domains of existence (dotted) of tensile solutions $y(x)$ to Problem (h, H). The boundary manifold Σ defined in Theorem 3.7 depends on k. For $k = 0$ (*Föppl model*) existence extends to all values $H < 0$. The surface load is uniform, $\nu = 1/3$, and $\varepsilon = 0.1$.

Analogues estimates for the arc $\rho_2(h)$ are somewhat more complicated but principally derivable from (3.17) and (3.22).

An extension of Pirner's mapping argument to Föppl's small finite deflection theory needs a few changes in the above analysis and has been elaborated in [16, 22] within a more general setting of two-point nonlinear boundary value problems. As was seen in the discussion of Problem (s, H), the set $Ex(s, H)$ alters significantly when the theory changes from Föppl's model to Reissner's model. This was a consequence of rn-solutions $g(z; s, S)$ to Problem (s, S) getting an *unbounded* derivative $\dot{g}(1; s, S) \to -\infty$ as $S \to 0+$ in Föppl's theory, which was seen by virtue of (3.12). The question whether a similar behavior occurs at $z = 0$ has a negative answer. For a proof, the superfunction $g_\infty(z)$ of Problem (s, S), $s = 0$, derived from (3.11) has to be replaced by a more appropriate one, for example by

$$g_\infty(z) := zS + \frac{2}{S^2} \int_0^1 k(z, t) \frac{P^2(t, \varepsilon)}{t^2} \, dt ,$$

which possesses a *bounded* derivative

$$\dot{g}_\infty(0) = S + \frac{2}{S^2} \int_0^1 (1 - t) \frac{P^2(t, \varepsilon)}{t^2} \, dt < +\infty .$$

Using (3.20) it becomes evident that $g(z; s, S)$ has a bounded derivative at $z = 0$, $s = 0$. As a consequence, the existence statement of Problem (h, S) expressed by Theorem 3.6 does not change substantially if the small finite deflection theory is considered.

However, Theorem 3.7 needs some modifications since the arc $\gamma_1(s)$ degenerates to $-\infty$ because $\dot{g}_1(1; s)$ now becomes unbounded for each $s \geq 0$, and because $\Gamma_1(S) \to -\infty$ as $S \to 0+$. Thus the separatrix Σ consists of the arc Σ'' alone. A suitable explicit representation of Σ'' is provided by

$$\Sigma'' = \{(h, H) \colon h = \rho_2^{-1}(H) = \Gamma_0[\Gamma_1^{-1}(H)],\ H \text{ real}\} .$$

The range of ρ_2^{-1} is the segment $(h_0, +\infty)$ with $h_0 := \Gamma_0(0)$. Therefore, the set

$$Ex(h, H) := \{(h, H) \colon h \leq \rho_2^{-1}(H),\ H \text{ real}\}$$

has to be recognized as the domain of existence of rn-solutions to Problem (h, H). In summary, the following final existence theorem holds.

THEOREM 3.8 [The existence statement of Problem (h, H) for the small finite deflection theory]. *Let $g_j(z; \cdot)$ and $\Gamma_j(S)$ be defined as in Theorems* 3.6 *and* 3.7. *Then the parameter range of Problem (h, H) is decomposed by Σ'' into complementary subsets of existence and nonexistence as follows*:

(a) *A unique rt-solution $g(z)$ exists if and only if $h < \rho_2^{-1}(H) := \Gamma_0[\Gamma_1^{-1}(H)]$ and $H \in \mathbb{R}$. The continuous arc ρ_2^{-1} is strictly increasing with $\lim_{H \to -\infty} \rho_2^{-1}(H) = h_0 := \Gamma_0(0)$ and $\lim_{H \to +\infty} \rho_2^{-1}(H) = +\infty$.*

(b) *At any boundary point $(h = \rho_2^{-1}(H), H)$ a unique rn-solution $g(z)$ is given by $g_0(z; S)$ and thus satisfies $g(0) = 0$ and $g(1) = S = \Gamma_1^{-1}(H)$.*

(c) *Outside the set $Ex(h, H)$ rn-solutions cannot exist.*

4. Wrinkle-free solutions of circular and annular membranes

Stability considerations limit the applicability of tensile and rn-solutions to engineering problems. According to Stein and Hedgepeth [42], buckling of a stretched membrane is termed 'wrinkling', and the criterion adopted in [42] for wrinkling is that in a membrane compressive principal stresses cannot occur. Hence the vanishing of the minimum principal stress is taken to be the condition for incipient wrinkling. Recently this criterion has been demonstrated by a stability analysis, even for physically nonlinear membrane theory, by Steigmann [41]. Since $\sigma_r \geq 0$ in the solutions discussed in the preceding sections, the condition of wrinkling is the vanishing of σ_θ. The same criterion was used also by Jahsman et al. [24] and by Nachbar [32] in their membrane analyses under point loads.

In his work on the Föppl circular membrane under uniform pressure, Dickey [10] observed from his numerical results that $\sigma_\theta(r)$ is monotone decreasing in $0 \leq r \leq a$. Hence, $\sigma_\theta(a) = 0$ is the wrinkling criterion. He found numerically that $\sigma_\theta(a) = 0$ for $S = 0.7292$, which implies that for $S > 0.7292$ the membrane is entirely in tension, while for $S < 0.7292$ circumferential compression occurs in some annulus $0 < c < r \leq a$. For the fixed edge problem $H = 0$, he found that $S \geq 0.7292$ for all ν, $0 \leq \nu < 1/2$.

It is seen that the values S and H separating stable from unstable solutions can easily be determined by solving the differential equations (2.1) or (2.16) (the ν-term omitted) for the boundary condition $y'(1) + y(1) = 0$, provided that $\sigma_\theta'(x) \leq 0$ holds for all $x \in [0, 1]$. This monotonicity was first proved for the circular membrane under certain variable loads $\bar{p}(x)$ by Weinitschke [45], both within the Föppl and the simplified Reissner theory. Set $z(x) := xy' + y$, then from (2.16)

$$x^3 z' = -x^3 y' - \frac{4}{k^2} x^2 \tilde{f}(x, y), \tag{4.1}$$

from which

$$x^3 z' = h(x) - xh'(x), \qquad h(x) = \frac{4}{k^2} \int_0^x t^2 \tilde{f}(t, y(t))\, dt \tag{4.2}$$

is easily derived. An elementary argument then shows that $z' \leq 0$ provided that $Q(x) > 0$ and $[x\bar{p}(x)]' \geq 0$. In that case z and therefore σ_θ is monotone decreasing. The same argument goes through for the Föppl membrane. The resulting stability limit curves are shown in Fig. 1.

A special result for annular membranes, but within the Föppl theory only, has also been established in [45]. It concerns the case $\bar{p} = 1$ and $s = 0$, for which we have the representation theorem (3.7). It is easy to derive a formula for $z = xy' + y$ from (3.7) which shows that z is monotone decreasing for all $x \in [\varepsilon, 1]$. Hence, the limit curves $S(\varepsilon)$ and $H(\varepsilon)$ separating the wrinkle-free solutions from the unstable ones are given by the condition $y'(1) + y(1) = 0$.

It was a major task to generalize the above results to circular and annular membranes under general boundary conditions, and to remove restrictions such as $(x\bar{p})' \geq 0$ or $\bar{p} = 1$ in the above results. This has been achieved in 1990 by Grabmüller [15] for the annular membrane and in 1991 by Beck and Grabmüller [4] for the circular membrane. We proceed to outline these new results.

We first observe that the transformation (3.4) can also be used in the case $\varepsilon = 0$. If we let $t := x^2$ and $g(t) := x^2 y(x)$, then Problem S reduces to

$$-\ddot{g}(t) = F(t, g)\,, \quad 0<t<1\,, \qquad g(0)=0\,, \qquad g(1)=S\,, \tag{4.3}$$

where a dot denotes differentiation with respect to t in this section. $F(t, g)$ is obtained from (3.5) by setting $\varepsilon = 0$. Since F is nonnegative, any rn-solution $g(t)$ of (4.3) must be concave. Furthermore, $g(t)$ satisfies an integral equation of the form (3.8), with $q(t) = St$, which can be utilized to extend differentiability of $g(t)$ to the end points $t = 0$ and $t = 1$. In particular, it can be shown that any solution of (4.3) satisfies $\dot{g}(0) = y(0) > 0$ and implies the correct boundary condition $y'(0) = 0$.

Let $g_j(t)$ be solutions of Problem S for $S = S_j$, $j = 1, 2$ with $S_2 > S_1 \geqslant 0$, and let $z(t) := g_2(t) - g_1(t)$. Then the integral representations for $g_j(t)$ can be used to prove that sign $\ddot{z}(t) =$ sign $z(t)$ and that $z(t) \geqslant 0$ and $\dot{z}(t) > 0$ holds for all $t \in [0, 1]$. In terms of the variables t and g, the circumferential stress can be written as

$$\sigma_\theta(t) = \frac{1}{4}\, Ek^2\left[2\dot{g}(t) - \frac{1}{t}\, g(t)\right], \quad 0 \leqslant t \leqslant 1\,. \tag{4.4}$$

It is seen that the rn-solution $g(t)$ of Problem S for $S = 0$ is not wrinkle-free. Indeed, by concavity we have $\dot{g}(1) < 0$ while $g(1) = 0$, implying $\sigma_\theta(1) < 0$. Consequently, a wrinkle-free solution must be an rt-solution satisfying $\dot{g}(t) > 0$ for all $t \in [0, 1]$. The function $\rho(t) := 2t\dot{g}(t) - g(t)$ obviously preserves the positivity of σ_θ except at $t = 0$. Let $g_j(t)$ be defined as above, then it follows from the monotonicity properties of z, $\dot{z}$ and $\ddot{z}$ that the corresponding functions $\rho_j(t)$ satisfy the inequality

$$\rho_2(t) - \rho_1(t) = t\dot{z}(t) + \int_0^t \tau\ddot{z}(\tau)\, \mathrm{d}\tau > 0\,, \quad 0 < t \leqslant 1\,. \tag{4.5}$$

Therefore, if $g(t)$ is wrinkle-free at a point (k, S), $S > 0$, then this property is preserved at any point (k, S_1) with $S_1 > S$. As the set

$$W := \{(k, S)\colon \inf_{0<t\leqslant 1} \rho(t; k, S) \geqslant 0\}\,, \tag{4.6}$$

is the domain of existence of wrinkle-free solutions, its boundary $\Gamma(k) = \inf_{S\geqslant 0} W$, $k > 0$ is well-defined.

In order to study $\Gamma(k)$, two maximum principles are needed. They can be proved under the following assumptions on the load: $p(t) \geqslant 0$ (or $p(t) \leqslant 0$) is measurable and bounded, $p > 0$ (or $p < 0$) holds on a subset of $[0, 1]$ of positive measure, $Q(t) \neq 0$ for all $t \in [t_0, 1]$, $0 < t_0 < 1$, and $(\mathrm{d}/\mathrm{d}t)Q^2(t)$ is piece-wise continuous for $t \in (0, 1)$, where $Q(t) = \int_0^{\sqrt{t}} \tau\bar{p}(\tau)\, \mathrm{d}\tau$ [15].

THEOREM 4.1. *Let $g(t)$ denote an rt-solution of Problem S, $S > 0$, then a local minimum $\rho(t_1) \leqslant 0$ cannot be attained at an interior point $t_1 \in (0, 1)$, unless $\dot{g}(1) \leqslant 0$.*

THEOREM 4.2. *Given $k_2 > k_1 > 0$, denote by $g_j(t) := g(t; k_j)$ the rn-solutions of Problem S for $S = S_j \geqslant 0$, $j = 1, 2$. Let $z(t) := g_2(t) - g_1(t)$. Then a local maximum $z(t_1) > 0$ cannot be attained at an interior point $t_1 \in (0, 1)$.*

According to Theorem 4.1, the stress component $\sigma_\theta(t)$ of a wrinkle-free solution $g(t)$ cannot vanish at an interior point of the membrane. Thus, positivity of $\sigma_\theta(t)$ is controllable via the positivity of the boundary value $\rho(1)$. This generalizes a similar result derived from (4.1) and (4.2), under the more restrictive assumptions $Q(t) > 0$ and $(\mathrm{d}/\mathrm{d}t)[t\bar{p}(t)] \geq 0$. Integral representations of $g(t)$ and $\dot{g}(t)$ give rise to the formula

$$\rho(1) \equiv \rho(1; k, S) = 2\dot{g}(1) - S = S - \frac{2}{k^2} \int_0^1 R[\tau, g(\tau)]\, \mathrm{d}\tau\,, \tag{4.7}$$

where

$$R(t, g) := 1 - \frac{g}{[g^2 + k^2P^2(t)]^{1/2}}\,, \qquad P(t) := 2\sqrt{t} \int_0^{\sqrt{t}} \tau\bar{p}(\tau)\, \mathrm{d}\tau\,.$$

R is related to F of (4.3) by $F(t, g) = R(t, g)/(k^2 t)$. It is seen from (4.7) that $\rho(1; k, S) \to +\infty$ for each fixed $k > 0$, as $S \to +\infty$. Since $\rho(1; k, 0) < 0$, the segment $[0, +\infty)$ is contained in the range of the mapping $S \mapsto \rho(1; k, S)$, for $S \geq 0$, $k > 0$. Hence the boundary $\Gamma(k)$ defined above can be obtained from the nonempty set

$$\Gamma := \{(k, S),\ k > 0,\ S \geq 0 \colon \rho(1; k, S) = 0\}\,, \tag{4.8}$$

by applying an implicit function theorem to $\rho(1; k, S) = 0$. At this point the continuity and strict monotonicity (4.5) of $\rho(1)$ is used. The set (4.8) then is the graph of a uniquely defined function $\Gamma\colon \mathbb{R}_+ \to \mathbb{R}$ satisfying $\rho(1; k, \Gamma(k)) = 0$ for all $k > 0$. The function $\Gamma(k)$ constitutes the finite boundary part of W provided $\dot{g}(1) > 0$. In view of (4.7), this condition holds for $S \geq \Gamma(k)$, $k > 0$.

Now Theorem 4.2 is applied to prove

THEOREM 4.3. *The mapping* $\Gamma(k)$ *is strictly decreasing and continuous*, $\lim_{k\to\infty} \Gamma(k) = 0$ *and* $\lim_{k\to 0+} \Gamma(k) = S^* > 0$ *exist.*

The following representation is an immediate consequence of (4.7)

$$\Gamma(k) = \frac{2}{k^2} \int_0^1 R[\tau, g(\tau)]\, \mathrm{d}\tau\,, \quad k > 0\,, \tag{4.9}$$

where $g(t)$ is the rt-solution satisfying $\rho(1) = 0$. Equation (4.9) shows that $\Gamma(k) \to 0$ as $k \to +\infty$. The limit for $k \to 0+$ can be obtained by applying l'Hospital's rule to (4.9). The result is

$$\Gamma(0) = S^* = \int_0^1 \frac{P^2(\tau)}{g^2(\tau)}\, \mathrm{d}\tau > 0\,. \tag{4.10}$$

S^* is the boundary of the Föppl model, so that rt-solutions are wrinkle-free for $S \geq S^*$. For uniform load $S^* = 0.7292$, as computed by Dickey [10].

It remains to consider the displacement problem. In terms of the variables t and g, the boundary conditions given in (2.3) and (2.16) for Problem H are

$$\begin{aligned} &2\dot{g}(1) - (1 + \nu)g(1) = H\colon && \text{Föppl}\,, \\ &2\dot{g}(1) - g(1) - \nu[g^2(1) + k^2P^2(1)]^{1/2} = H\colon && \text{simplified Reissner}\,. \end{aligned} \tag{4.11}$$

Introducing $\rho(1)=0$ and making use of (4.7) the domain of wrinkle-free solutions for Problem H is bounded by the graph of

$$\begin{aligned} H(k) &= -\nu S^* , && k=0: \text{ Föppl}, \\ H(k) &= -\nu[\Gamma^2(k)+k^2P^2(1)]^{1/2}, && k>0: \text{ simplified Reissner}. \end{aligned} \tag{4.12}$$

An asymptotic form of $H(k)$ is $0 \leqslant -H(k) - \nu P(1)k = o(1)$ as $k \to \infty$, which follows from Theorem 4.3. The limit curves $\Gamma(k)$ and $H(k)$ are presented in Fig. 1 for the case of uniform load and $\nu = 1/3$. It is worth noting that there is a strong dependence on ν. In fact, (4.12) shows that $H(k)=0$ for all $k \geqslant 0$ if $\nu = 0$.

We now discuss the problem of wrinkle-free solutions for *annular membranes*, solved by Grabmüller [15] under similar assumptions on the surface load as for the circular membrane. The basic boundary value problems have been formulated in equations (3.1)–(3.3). The Schwerin transformation (3.4) is used to allow the application of concavity and monotonicity arguments. A maximum principle is then established which shows that the positivity of $\sigma_\theta(x)$ is controllable via the values taken at the boundary $x=\varepsilon$ and $x=1$. We then examine in the set of boundary data $s \geqslant 0$, $S \geqslant 0$ the manifolds $\sigma_\theta(\varepsilon)=0$ and $\sigma_\theta(1)=0$ (keeping the parameters k and ε fixed). They determine the boundary of the domain $W(s,S)$ of wrinkle-free solutions of Problem (s, S).

In terms of the variables $t=(x^2-\varepsilon^2)/(1-\varepsilon^2)$, $g(t)=\omega^4x^2y(x)$ we have the annular membrane boundary value problems (3.5) and (3.6), replacing z by t, as above. Let $g_j(t)$ denote any two rn-solutions of Problem (s_j, S_j), $j=1,2$, and let $z(t) := g_2(t)-g_1(t)$, then we have again monotonicity $z(t) \geqslant 0$, $t \in [0,1]$, $\ddot{z}(t) \geqslant 0$, $t \in (0,1)$, provided that $s_2 \geqslant s_1 \geqslant 0$ and $S_2 \geqslant S_1 \geqslant 0$. Furthermore, $\dot{z}(t) \geqslant 0$, $t \in [0,1]$, provided that $s_2 > s_1 \geqslant 0$ and $S_2 = S_1 \geqslant 0$. The function

$$\rho(t) := \omega(t+\gamma)\sigma_\theta(t)(4/Ek^2) = 2(t+\gamma)\dot{g}(t) - g(t), \tag{4.13}$$

where $\gamma = \varepsilon^2/(1-\varepsilon^2)$, preserves both the regularity and positivity of σ_θ, thus

$$W(s,S) = \{(s,S): \inf_{0<t\leqslant 1} \rho(t;s,S) \geqslant 0\}. \tag{4.14}$$

As in the case of the circular membrane, it is observed that an rn-solution $g(t)$ of Problem (s, S) is not wrinkle-free if $s=S=0$. Clearly a wrinkle-free solution $g(t)$ must satisfy $\dot{g}(t)>0$ for all $t \in [0,1]$, which follows from (4.13) noting that $g(1)>0$, and that $\dot{g}(0) \geqslant \dot{g}(t) \geqslant \dot{g}(1)$ by concavity.

In order to derive monotonicity properties for $\rho(t)$, the integral equation (3.8) is employed. Together with the monotonicity of $z = g_2 - g_1$, the following inequalities analogous to (4.5) are obtained for $\rho_j(t) = \rho(t; s_j, S_j)$, $j=1,2$

$$\rho_2(t) - \rho_1(t) = (t+2\gamma)\dot{z}(t) + \int_0^t \tau\ddot{z}(\tau)\,\mathrm{d}r \geqslant 0, \quad t \in [0,1], \tag{4.15}$$

provided that $s_2 = s_1 \geqslant 0$ and $S_2 > S_1 \geqslant 0$. On the other hand, if $s_2 > s_1 \geqslant 0$ and $S_2 = S_1 \geqslant 0$, then

$$\rho_2(t) - \rho_1(t) = 2(t+\gamma)\dot{z}(t) - z(t) \leqslant 0, \quad t \in [0,1]. \tag{4.16}$$

Next the structure of the boundary $\Gamma(s)=\inf_S W(s,S)$, $s\geq 0$, is studied under the assumption that $p(x)$ is measurable and positive on $\varepsilon\leq x\leq 1$ in the sense defined above for the circular membrane. Theorem 4.1 holds for any rn-solution of Problem (s, S), $S>0$ ($0\leq t\leq 1$ represents the interval $\varepsilon\leq x\leq 1$). Thus, positivity of σ_θ is here controlled by the boundary values $\rho(0)$ *and* $\rho(1)$, given from (4.13) by

$$\rho(0)=2\gamma\dot{g}(0)-s\,,\qquad \rho(1)=2(1+\gamma)\dot{g}(1)-S\,. \tag{4.17}$$

If both $\rho(0;s,S)$ and $\rho(1;s,S)$ are zero for $s=s_p$, $S=S_p$, then (s_p,S_p) is called a *switch point*. The integral representation for $\rho(t)$, evaluated at $t=0$ and $t=1$, supplies two linear algebraic equations for calculating switch points. The result is

$$\begin{aligned} s_p&=2\gamma\int_0^1(1+2\gamma+\tau)F[\tau,g(\tau)]\,\mathrm{d}\tau>0\,,\\ S_p&=2(1+\gamma)\int_0^1(2\gamma+\tau)F[\tau,g(\tau)]\,\mathrm{d}\tau>0\,. \end{aligned} \tag{4.18}$$

It can be shown that there exists exactly one switch point for any given $k>0$ and $\varepsilon>0$. Here again the monotonicity of the function $z(t)$ is crucial in the proof. It is easy to compute the switch point numerically. One simply solves the differential equation $-\ddot{g}(t)=F(t,g)$ subject to the boundary conditions (4.17) setting $\rho(0)=\rho(1)=0$ and $s=g(0)$, $S=g(1)$.

Physically, one expects wrinkling to occur if s at the inner edge is sufficiently large. On the other hand, no wrinkling should occur if S at the outer edge is sufficiently large. Indeed, from an integral representation for $\rho(t)$ one can show rigorously that $\rho(0;s,S)\to-\infty$ for each $S\geq 0$ as $s\to+\infty$, and $\rho(1;s,S)\to+\infty$ for each $s\geq 0$ as $S\to+\infty$. Hence the ranges of the mappings $s\mapsto\rho(0;s,S)$ and $S\mapsto\rho(1;s,S)$ are such that the nonempty sets

$$\Gamma_j=\{(s,S):\ \rho(j;s,S)=0\}\,,\quad j=0,1 \tag{4.19}$$

give rise, via the implicit function theorem, to a pair of uniquely defined functions $\Gamma_0^{-1}:\ \bar{\mathbb{R}}_+\to\mathbb{R}$ and $\Gamma_1:\ \bar{\mathbb{R}}_+\to\mathbb{R}$ satisfying

$$\rho(0;\Gamma_0^{-1}(S),S)=0\quad\text{for } S\geq 0\quad\text{and}\quad\rho(1;s,\Gamma_1(s))=0\quad\text{for } s\geq 0\,. \tag{4.20}$$

With $s^*:=\Gamma_0^{-1}(0)$ and $S^*:=\Gamma_1(0)$, it follows that $D(\Gamma_0)=\mathrm{Range}(\Gamma_0^{-1})=[s^*,+\infty)$ and $\mathrm{Range}(\Gamma_1)=[S^*,+\infty)$ because the mapping $\Gamma_0^{-1}:\ \bar{\mathbb{R}}_+\to D(\Gamma_0)$ is one-to-one and the mappings $\Gamma_0:\ D(\Gamma_0)\to\mathbb{R}$ and $\Gamma_1:\ \bar{\mathbb{R}}_+\to\mathbb{R}$ are strongly increasing and continuous; furthermore, $\lim_{s\to\infty}\Gamma_j(s)=+\infty$, $j=0,1$.

The domain $W(s,S)$ of wrinkle-free solutions of Problem (s, S) can now be determined. In view of (4.17), the condition $\dot{g}(1)>0$ holds for $S\geq\Gamma_1(s)$ and $s\geq 0$. Thus Theorem 4.1 applies which shows that the boundary Γ of $W(s,S)$ is part of the arcs Γ_0 and Γ_1. More precisely, we have:

THEOREM 4.4. *There exists a unique switch point* (s_p,S_p). *The parameter domain of wrinkle-free solutions of Problem* (s,S) *is the set* (*see Fig.* 4)

$$W(s,S)=\{(s,S):\ S\geq\Gamma_1(s)\ \text{for}\ 0\leq s\leq s_p,\ S\geq\Gamma_0(s)\ \text{for}\ s_p\leq s<\infty\}\,.$$

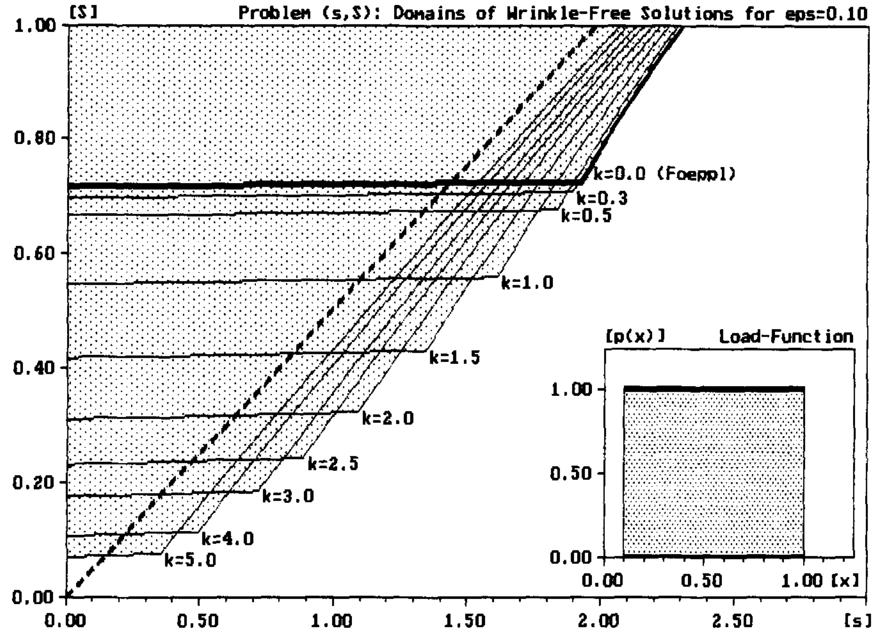

Fig. 4. Wrinkle-free solutions of Problem (s, S) only exist within the dotted domains. At the boundary curves that asymptotically approach the straight line $S=(1+\varepsilon^2)s/2$ (bold dotted), the circumferential stress σ_θ vanishes at one of the edges of the annulus, i.e., at the outer edge on the horizontal arcs, and at the inner edge on the inclined arcs. The surface load is uniform, $\nu = 1/3$, and $\varepsilon = 0.1$.

Next we consider problems involving displacement data in the boundary conditions. Following Theorem 3.4, the images of mappings defined via the displacement boundary operators B_j, $j=0,1$, were seen to be the domains where rn-solutions of the respective Problems (s, H), (h, S) and (h, H) exist. Now it becomes obvious that the subdomain $W(s,S) \subseteq Ex(s,S)$ of wrinkle-free solutions is mapped onto a corresponding subdomain in the boundary-parameter ranges (s,H), (h,S) and (h,H). In particular, we are interested in finding the images of the arcs $\Gamma_0(s)$ and $\Gamma_1(s)$. To this end we simply have to combine (4.17), setting $\rho(0)$ and $\rho(1)$ equal zero, with the displacement boundary operators B_j defined following (3.6). Hence, we have, with $\tilde{\nu} := (1-\varepsilon^2)\nu/2$ and $\Theta_1 := \Theta(1)$

$$\text{on } \Gamma_0: \quad h_0(s) = \varepsilon^2 \dot{g}(0) - \mu s = \frac{\varepsilon^2}{2\gamma}\, s - \mu s = -\tilde{\nu} s\,, \tag{4.21}$$

$$\text{on } \Gamma_1: \quad H_1(s) = \begin{cases} \dot{g}(1) - \mu g(1) = \dfrac{1}{2(1+\gamma)}\, S - \mu S = -\tilde{\nu}\Gamma_1(s)\,, \\ \dot{g}(1) - \dfrac{1-\varepsilon^2}{2}\,[g(1) + \nu\sqrt{g^2(1)+\Theta_1^2}] = -\tilde{\nu}(\Gamma_1^2(s)+\Theta_1^2)^{1/2}\,, \end{cases} \tag{4.22}$$

for the Föppl and simplified Reissner theory. If the boundary condition $B_1[g] = H$ is rewritten in terms of $\Gamma_0(s)$ we find

$$H_0(s) = \begin{cases} D_1(s) - \mu\Gamma_0(s): & \text{Föppl}\,, \\ D_1(s) - \mu\Gamma_0(s) - \tilde{\nu}\Gamma_0(s)[(1+\Theta_1^2/\Gamma_0^2(s))^{1/2} - 1]: & \text{simpl. Reissner}\,, \end{cases} \tag{4.23}$$

where $D_1(s) = \dot{g}(1)$, $g(t)$ being the solution of Problem (s, S) for $S = \Gamma_0(s)$.

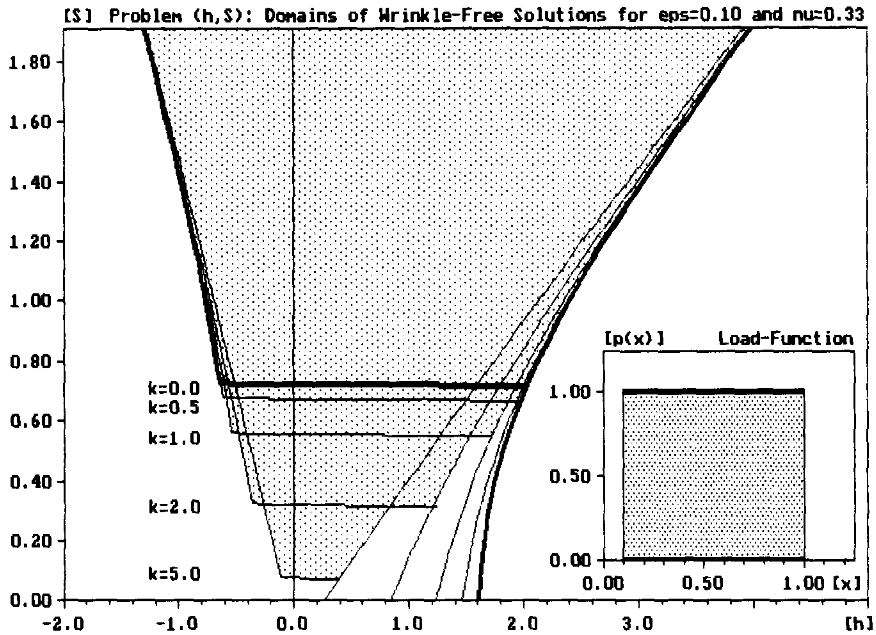

Fig. 5. The domains (dotted) of wrinkle-free solutions $y(x)$ of Problem (h, S) for various k. Only minor differences are found between the results for $k = 0$ (*Föppl model*) and $k > 0$ (*Reissner model*). The surface load is uniform, $\nu = 1/3$, and $\varepsilon = 0.1$.

The set $W(s, H)$ of wrinkle-free solutions of Problem (s, H) can thus be written as follows:

$$W(s, H) = \{(s, H): H \geq H_1(s) \text{ for } 0 \leq s \leq s_p,\ H \geq H_0(s) \text{ for } s_p \leq s < \infty\}\ .$$

We note that $H_1(s) = 0$ for all k if $\nu = 0$. Therefore, no wrinkle-free solution can exist for $H < 0$.

Changing to a parametrization with respect to S, the domain $W(h, S)$ of wrinkle-free solutions of Problem (h, S) can similarly be determined. Clearly (4.21) can be written as $h_0(S) = -\tilde{\nu}\Gamma_0^{-1}(S)$, in (4.22) $\Gamma_1(s)$ is simply replaced by S. The domains of wrinkle-free solutions of Problem (h, S) are shown in Fig. 5. In the case of Problem (h, H) the boundary of the corresponding domain $W(h, H)$ decomposes into two connecting arcs which can be expressed via the function H_0, H_1 defined in (4.22) and (4.23) and a function $h_1(S)$ defined for describing $W(h, S)$; for details see [15].

5. Curved membranes of revolution

We begin by discussing some recent work for shallow membranes for small finite deflections (Föppl theory), obtained by Baxley [2] and Dickey [11]. The governing equation is (1.7). For a shallow spherical cap under normal pressure $p_s = 0 = p_H$ and $z \doteq z_0(1 - r^2)$, approximately. Since $z' = \sin \varphi$, this implies $1 - \cos \varphi \doteq 2z_0 r^2$ in (1.7). Introducing dimensionless variables as for the circular membrane, equation (1.7) can be reduced to the form

$$Ly = \frac{2}{y^2}\, Q^2(x) - \lambda_0^2\,, \qquad 0 < x < 1\,, \qquad y'(0) = 0\,, \tag{5.1}$$

where $x = r/a$, L and Q are defined in (2.1), and $\lambda_0 = (4Ea^2d/p_0)^{1/3}/R\sqrt{2}$ is a geometry-load parameter proportional to the height z_0 of the spherical cap, R is the radius of the sphere. Applying the transformation (3.4) with $t = x^2$ and $g(t) = x^2y(x)$, we obtain the problem formulated in [2] for uniform load and fixed edge, namely Problem H for $H = 0$, that is (apart from notation)

$$\begin{aligned} &\ddot{g}(t) = -\frac{t^2}{g^2} + \lambda^2, \qquad 0 < t < 1, \\ &g(0) = 0, \qquad 2\dot{g}(1) - (1+\nu)g(1) = 0. \end{aligned} \tag{5.2}$$

It is observed in [2] that the right-hand side of the differential equation is nondecreasing in g, at least for tensile solutions $g > 0$, but that the coefficients in the boundary condition at $t = 1$ do not have the right signs for a classical existence theorem to apply. However, it was shown in [17], how this difficulty can be overcome by the well-known maximum principle of Hopf. Hence, applying the technique of [17], the *uniqueness* of positive solutions of the boundary value problem (5.2) follows, even for variable load and more generally for Problem S and Problem H, in our terminology. In order to prove *existence*, Baxley substitutes $u = 1/t$, reduces (5.2) to a boundary value problem on $1 \leq u < \infty$, and applies an existence theorem valid for problems of the type

$$\begin{aligned} &w''(u) = f(u, w, w'), \qquad a \leq u < \infty, \\ &a_0w(a) - a_1w'(a) = A, \qquad a_0 > 0, \quad a_1 \geq 0, \end{aligned}$$

where f must satisfy a number of smoothness and monotonicity conditions. We summarize the results as

THEOREM 5.1 (Baxley). *Suppose* $(1+\nu)/2 < \rho < 1$, *then Problem* (5.2) *has at most one solution satisfying* $g(t)/t^\rho \to 0$ *as* $t \to 0+$. *A positive solution* $g(t)$ *exists for all* λ, *which has the properties*

$$\frac{g(t)}{t} < \frac{1}{2\lambda}, \qquad g(t) \geq \alpha t(\beta - t), \qquad \frac{g(t)}{t}\ \textit{decreasing}, \tag{5.3}$$

for $0 < t \leq 1$ *and some constants* $\alpha > 0$, $\beta > 1$. *Furthermore*, $\lim(g(t)/t)$ *exists for* $t \to 0+$.

It appears that the existence part of the theorem can also be proved by transforming (5.2) into an integral equation and applying Schauder's fixed point theorem. Indeed, this is the technique employed by Dickey [11] in his work on shallow shells for surfaces generated by rotating the curve

$$z = z(r) = C\left(1 - \left(\frac{r}{a}\right)^\gamma\right), \qquad 0 \leq r \leq a, \tag{5.4}$$

for a constant $C > 0$ and a shape factor $\gamma > 1$. The case $\gamma = 2$ and C sufficiently small corresponds to the spherical cap (5.1). Given the radial stress or the radial displacement at the edge $r = a$, we obtain Problems S and H in the dimensionless form

$$Ly = \frac{2}{y^2} Q^2(x) - \lambda_0^2 x^{2\gamma-4} , \qquad \lambda_0 = \frac{1}{\sqrt{2}} \left(\frac{4EC^2\gamma^3 d}{p_0 a^4} \right)^{1/3} ,$$
$$y'(0) = 0 , \qquad y(1) = S \quad \text{or} \quad y'(1) + (1-\nu)y(1) = H . \tag{5.5}$$

The technique of solving (5.5) employed in [11] is quite similar to the case $\lambda_0 = 0$, as discussed in detail in Section 2. The boundary value problem is transformed into an integral equation of type $y = Ty$, then the operator T is shown to have properties such that the Schauder fixed point theorem can be applied. The result is, proved in [11] for $Q = 1$ (uniform load):

THEOREM 5.2 (Dickey). *Set* $D = \lambda_0^2 4^{-5/3}/[\gamma(\gamma-1)]$ *and assume* $\gamma > 1$, $S > D$. *Then Problem S has a solution* $y(x)$, *which is positive for* $x \in [0, 1]$.

Although uniqueness is not discussed in [11], it is obvious that a standard uniqueness argument applies to (5.5) for any $S > 0$, so that positive solutions of Problem S are unique. Problem H, $H = 0$, is solved by interpolation as in [10], showing that $u(a)$ changes sign for solutions of Problem S in a range of S covered by Theorem 5.2. In addition, D must be sufficiently small. The result is, again for $Q = 1$:

THEOREM 5.3 (Dickey). *Assume* $\gamma > 1$ *and* D *is sufficiently small, then Problem H, for* $H = 0$, *has a positive solution* $y(x)$.

In the special case $\gamma = 2$, Theorem 5.1 shows that a smallness assumption on D is unnecessary. However, it is proved in [11] that if $\gamma = 4/3$, Problem H for $H = 0$ has no positive solution unless D is sufficiently small. This particular case is amenable to a phase plane analysis, from which the following conclusions are drawn in [11]. If $\lambda^2 \leq (64/3)^{1/3}$, where $\lambda^2 = c\lambda_0^2$, then Problem S has a unique rt-solution $y(x)$ for all $S > 0$, and Problem H has a unique rt-solution $y(x)$ for all H. If $\lambda^2 > (64/3)^{1/3}$ then Problem S has no tensile solution unless $S > S_0$, and Problem H has no rt-solution unless $H > H_0$ where S_0 and H_0 are positive numbers. In particular, there is no (rotationally symmetric) solution of Problem H for $H = 0$ in the case $\gamma = 4/3$ and $\lambda^2 > (64/3)^{1/3}$.

Finally, we present some work from the Thesis of J. Arango [1], which concerns deformation of curved membranes with finite rotations under uniform *normal pressure* $p_n = q$. The basic equation is (1.6), where $p_s = 0$. From equations (1.4) we have $p_H = q \sin \Phi$, which means that V in (1.6) is not a given quantity, but rather depends on the solution Φ. In fact, equilibrium in the z-direction implies

$$\frac{\mathrm{d}(rV)}{\mathrm{d}s} + rp_V = 0 = \frac{\mathrm{d}(rV)}{\mathrm{d}s} - r(s)q \cos \Phi . \tag{5.6}$$

Following the notation in Clark et al. [8], we substitute

$$M = rS_\phi \cos \Phi , \qquad N = rS_\phi \sin \Phi , \qquad H = rS_\theta , \tag{5.7}$$

where S_ϕ and S_θ are pseudo stress resultants which measure the tension in the deformed membrane per unit length of the undeformed membrane. Then the basic equations (1.6) and (5.6) can also be written as an equivalent first order system of two equilibrium equations and one compatibility equation [8]

$$r\frac{\mathrm{d}M}{\mathrm{d}s} - H + r^2q\sin\Phi = 0\,, \qquad r\frac{\mathrm{d}N}{\mathrm{d}s} - r^2q\cos\Phi = 0\,,$$
$$r\frac{\mathrm{d}H}{\mathrm{d}s} - M - Edr(\cos\Phi - \cos\varphi) = 0\,. \tag{5.8}$$

Assuming $0 < s \le L$, tensile solutions must satisfy $S_\phi(s) > 0$ for $0 < s \le L$. In addition, rotations are restricted to $0 \le \Phi(s) < \pi/2$, so that $M > 0$ and $N > 0$ for $0 < s \le L$. For wrinkle-free solutions we also have $H \ge 0$. Observing the relations

$$\cos\Phi = M/(M^2 + N^2)^{1/2}\,, \qquad \sin\Phi = N/(M^2 + N^2)^{1/2}\,,$$

the system (5.8) can be re-written in the dimensionless form

$$\begin{aligned} x'(t) &= z(t)/\rho(t) - \rho(t)y(t)/Q(x, y)\,, \\ y'(t) &= \rho(t)x(t)/Q(x, y)\,, \quad 0 < t < 1\,, \\ z'(t) &= x(t)/\rho(t) + e[-\rho'(t) + x(t)/Q(x, y)]\,, \end{aligned} \tag{5.9}$$

where $t = s/L$, $\rho = r/L$, $(x, y, z) = (M, N, H)/(L^2q)$, $Q(x, y) = (x^2 + y^2)^{1/2}$ and $e = Ed/qL$. If the membrane is closed at the apex, one has $\rho(0) = 0$ and $\rho'(0) = 1$. Any regular solution of (5.9) must satisfy $x(0) = y(0) = z(0) = 0$. If the membrane has a circular inner edge at $r = r_0 > 0$, then $\rho(t) > 0$ for $0 \le t \le 1$ and $x(0)$, $y(0)$ or $x(0)$, $z(0)$ can be prescribed at the inner edge. In both cases the geometry is restricted by the assumption $\rho'(t) > 0$ for $0 \le t \le 1$. At the outer edge, a boundary condition

$$B(x(1), y(1), z(1)) = m \tag{5.10}$$

is imposed, where $B\colon \mathbb{R}^3 \to \mathbb{R}$ is a given function and m is a given number. The problem is then to find sufficient conditions on the boundary conditions, for certain physically meaningful B, such that equations (5.9), (5.10) have a tensile solution, and also to formulate conditions for uniqueness, and for wrinkle-free solutions.

The method of proof in [1] is essentially the shooting method. A set of *admissible solutions* $X(t, r) = (x(t, r), y(t, r), z(t, r))$, $0 \le t \le 1$, is introduced that satisfy the conditions at $t = 0$ stated above and the differential equations (5.9), and that depend on a shooting parameter r. The existence of solutions of (5.9) and (5.10) is then equivalent with the existence of zeros of the shooting function

$$F(r) := B(X(1, r)) - m\,.$$

Under appropriate assumptions on $F(r)$ there is a unique zero which yields uniqueness of the solution of the boundary value problem. It is clear that one of the main difficulties is to establish the existence of admissible solutions $X(t, r)$ in the interval $0 \le t \le 1$. Fortunately, the nonlinear nonnegative terms x/Q and y/Q in (5.9) are bounded by unity. This fact, together with some monotonicity properties of the solutions $X(t, r)$, allows one to extend the local existence of $X(t, r)$ near $t = 0$, which is guaranteed by classical theory, to a global existence in the whole interval $0 \leq t \leq 1$. The details are quite involved and cannot be discussed here. Some representative results of [1] for a membrane of revolution closed at the apex are contained in the next two theorems. In analogy to flat and shallow membranes two boundary value problems are defined by prescribing the radial stress or the radial displacement at the boundary $t = 1$ (see (5.7))

$$\begin{aligned} &\text{Problem S:} \quad B(X) = x(1) , \\ &\text{Problem H:} \quad B(X) = z(1) - \nu[x^2(1) + y^2(1)]^{1/2} . \end{aligned} \tag{5.11}$$

Let

$$\bar{\rho}(t) = -\int_1^t \frac{1}{\rho(\tau)} \, \mathrm{d}\tau , \qquad \rho'(t) = 1 - t\hat{\rho}(t) , \quad t > 0 ,$$

then the assumptions on $\rho(t)$ imply that $\bar{\rho} > 0$, $\hat{\rho} > 0$ for $0 < t \leq 1$.

THEOREM 5.4 (Arango). *Problem S has a tensile solution for all $m > 0$ satisfying*

$$m > m_0 := e \int_0^1 \hat{\rho}(t) t \sinh(\bar{\rho}(t)) \, \mathrm{d}t > 0 .$$

Problem H has a tensile solution for all $m > 0$ satisfying

$$m > m_1 := \sup\left\{ e \int_0^1 \hat{\rho}(t) t \cosh(\bar{\rho}(t)) \, \mathrm{d}t, \int_0^1 [\cosh(\bar{\rho}(t)) - \sinh(\bar{\rho}(t))][\rho(t) + et\hat{\rho}(t)] \, \mathrm{d}t \right\} > 0 .$$

These solutions of Problems S and H are unique if

$$1 + (\rho'(t))^2 + \rho(t)\rho''(t) \geq 0 , \quad 0 \leq t \leq 1 . \tag{5.12}$$

The conditions $m > m_0$ and $m > m_1$ are perhaps overly restrictive. On the other hand, Theorem 5.4 is valid for quite arbitrary membranes of revolution. The results of Dickey show that some particular geometries do require restrictions on m, excluding $m = 0$ in Problem H. The above results simplify considerably for the circular membrane problem under *normal* pressure, which in the Reissner theory of finite rotations is different from the corresponding problem under *vertical* pressure.

THEOREM 5.5 (Arango). *In the case of a circular membrane $\rho'(t) \equiv 1$, Problem S has a unique tensile solution for all $m \geq 0$. Problem H has a unique tensile solution for all $m \geq 1/2$.*

While the result for Problem S is best possible, Problem H for a fixed edge ($m = 0$) is not covered by Theorem 5.5.

Existence and uniqueness results for membranes of revolutions with a circular opening have also been obtained in [1]. Depending on which boundary data are prescribed at $t = 0$ and $t = 1$, there results a variety of boundary value problems. As an example, we briefly discuss results corresponding to Problems (s, S) and (s, H) defined in Section 3. Suppose $x_0 \geq 0$, $y_0 \geq 0$ are such that $x_0^2 + y_0^2 > 0$, then one has

THEOREM 5.6 (Arango). *Let* $B(X) = x(1)$ *and* $m \geqslant 0$. *Then the boundary value problem* (5.9), (5.10) *has a tensile solution. This solution is unique, if* $\rho(t)$ *satisfies* (5.12). *Let* $B(X)$ *be defined as in* (5.11), *Problem H, and* $m \geqslant \rho^2(1)$, *then* (5.9), (5.10) *has at least one tensile solution.*

In the case of a flat annular membrane $\rho(t) = t$, the condition $m \geqslant \rho^2(1)$ can be improved to $m \geqslant 1/3$.

The above conditions do not cover the whole range of physically meaningful boundary data. Separation curves, as in Section 3, separating exactly the domains of existence and nonexistence of tensile solutions or establishing the domain of wrinkle-free solutions for the various boundary value problems have yet to be found. For the circular and annular membrane it is known that $S_\theta(t) \geqslant 0$ can be controlled by the boundary data as in Section 4. The following result was proved in [1]: An admissible solution of equations (5.9) is wrinkle-free if and only if $z(0) \geqslant 0$ and $z(1) \geqslant 0$.

The problem of curved membranes under variable *vertical* load has apparently not been investigated as yet (see a forthcoming thesis by A. Beck). In closing, we would also like to stress that the results of Sections 2–4 concerning finite rotation problems have all been derived under the simplifying assumption that the term $\nu r p_s$ in (1.6) is ignored. It remains to be seen whether the qualitative results concerning existence, uniqueness and wrinkle-free solutions change significantly if that term is included in the analysis. Numerical calculations in [45] show that quantitative differences are not negligible for larger values of k.

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References

1. J. Arango, Existenz und Eindeutigkeitsaussagen über Lösungen von Randwertproblemen der nichtlinearen Membrantheorie, Dissertation (Ph.D. Thesis), Faculty of Science I, Univ. Erlangen-Nürnberg, July 1989, 121 pp.
2. J.V. Baxley, A singular nonlinear boundary value problem: Membrane response of a spherical cap. *SIAM J. Appl. Math.* 48 (1988) 497–505.
3. A. Beck, Die Reissnersche Kreismembran: Existenz und Eindeutigkeit von positiven Lösungen, Diploma-Thesis, Faculty of Science I, University of Erlangen-Nürnberg, April 1989, 63 pp.
4. A. Beck and H. Grabmüller, Wrinkle-free solutions of circular membrane problems, *J. Appl. Math. and Phys.* (ZAMP) (in press).
5. E. Bromberg and J.J. Stoker, Non-linear theory of curved elastic sheets. *Quart. Appl. Math.* 3 (1945) 246–265.
6. A.J. Callegari and E.L. Reiss, Nonlinear boundary value problems for the circular membrane. *Arch. Rat. Mech. Anal.* 31 (1968) 390–400.
7. P. Ciarlet, *Plates and Functions in Elastic Multi-Structures, an Asymptotic Analysis.* Masson, Paris and Springer-Verlag, Berlin (1990).
8. R.A. Clark and O.S. Narayanaswamy, *Nonlinear membrane problems for elastic shells of revolution.* In: D. Muster (ed.), *Proc. Sympos. Theory of Shells* (L.H. Donnell Anniversary Volume). Univ. of Houston Press, Houston (1967) pp 80–110.
9. L. Collatz, *Funktionalanalysis und Numerische Mathematik.* Springer, Berlin–Göttingen–Heidelberg (1964).
10. R.W. Dickey, The plane circular elastic surface under normal pressure. *Arch. Rat. Mech. Anal.* 26 (1967) 219–236.

11. R.W. Dickey, Rotationally symmetric solutions for shallow membrane caps. *Quart. Appl. Math.* 47 (1989) 571–581.
12. L.H. Donnell, A new theory for the buckling of thin cylinders under axial compression and bending. *Trans. ASME* 56 (1934) 795–806.
13. A. Föppl, *Vorlesungen über Technische Mechanik*, Vol. 5. R. Oldenbourg, München (1907).
14. H. Grabmüller, Monotone methods for scalar second order boundary value problems with decreasing discontinuous nonlinearities and applications to annular membrane problems, Inst. Angew. Math., Univ. Erlangen-Nürnberg, Report 129, April 1988.
15. H. Grabmüller, Wrinkle-free solutions in the theory of annular elastic membranes. *J. Appl. Math. and Phys.* (ZAMP) 42 (1991) 783–805.
16. H. Grabmüller, Positive solutions and monotone methods for second order differential equations with discontinuous nonlinearity and one-sided nonlinear boundary condition, Inst. Angew. Math., Univ. Erlangen-Nürnberg, Report 130, April 1988.
17. H. Grabmüller and E. Novak, Nonlinear boundary value problems for the annular membrane: a note on uniqueness of positive solutions. *J. Elasticity* 17 (1987) 279–284.
18. H. Grabmüller and E. Novak, Nonlinear boundary value problems for the annular membrane: new results on existence of positive solutions. *Math. Meth. Appl. Sci.* 10 (1988) 37–49.
19. H. Grabmüller and R. Pirner, Positive solutions of annular elastic membrane problems with finite rotations. *Studies in Appl. Math.* 77 (1987) 223–252.
20. H. Grabmüller and R. Pirner, Existence theorems for some boundary value problems in the nonlinear theory of annular elastic membranes, *European J. of Appl. Math.* (in press).
21. H. Grabmüller and H.J. Weinitschke, Finite displacements of annular elastic membranes. *J. Elasticity* 16 (1986) 135–147.
22. H. Grabmüller, Monotone methods for second order differential equations with decreasing nonlinearity and nonlinear boundary conditions, SFB 123 Stochastische Mathematische Modelle, Universität Heidelberg, Preprint Nr. 467, June 1988.
23. H. Hencky, Über den Spannungszustand in kreisrunden Platten. *J. Math. Phys.* 63 (1915) 311–317.
24. W.E. Jahsman, F.A. Field and A.M.C. Holmes, Finite deformations in a prestressed, centrally loaded, circular elastic membrane. In: *Proc. 4th U.S. National Congress of Applied Mechanics*, ASME, New York (1962).
25. R. Kao and N. Perrone, Large deflections of axisymmetric circular membranes. *Int. J. Solids & Structures* 7 (1971) 1601–1612.
26. Th. von Kármán, *Festigkeitsprobleme im Maschinenbau*. Encyklopädie der Mathematischen Wissenschaften, Vol. 4/4 (1910) pp. 311–385.
27. H.B. Keller and E.L. Reiss, Iterative solutions of the non-linear bending of circular plates. *Comm. Pure Appl. Math.* 11 (1958) 273–292.
28. G.R. Kirchhoff, Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes. *J. Reine u. Angew. Math.* 56 (1859) 285–313 (also *Vorlesungen über Math. Physik, Mechanik*, 2nd edn, Leipzig (1877)).
29. W.T. Koiter, The intrinsic equations of shell theory with some application. In: S. Nemat-Nasser (ed.), *Mechanics Today* 5 (E. Reissner Anniversary Volume). Pergamon Press (1980) pp. 139–154.
30. A. Libai and J.G. Simmonds, *The Nonlinear Theory of Elastic Shells: One Spatial Dimension*. Academic Press Inc., Boston (1988).
31. K. Marguerre, Zur Theorie der gekrümmten Platte großer Formänderung. In: *Proc. 5th Intern. Congr. Appl. Mech.* (1938) pp. 93–101.
32. W. Nachbar, Finite deformation of membranes and shells under localized loading. In: F.I. Niordson (ed.), *Theory of Thin Shells*. IUTAM Symposium, Copenhagen 1967. Springer-Verlag, Berlin/Heidelberg/New York (1969) pp. 176–211.
33. E. Novak, On the convergence of interpolated iteration methods. *SIAM J. Math. Anal.* 19 (1988) 1174–1182.
34. A. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*. Academic Press, New York/London (1973).
35. R. Pirner, Randwertprobleme der nichtlinearen Membrantheorie: Über Existenz und Nichtexistenz positiver Lösungen. Diploma-Thesis, Faculty of Science I, Univ. Erlangen-Nürnberg, April 1987, 82 pp.
36. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1967).
37. E. Reissner, On axisymmetrical deformations of thin shells of revolution. In: *Proc. Symp. Appl. Math.*, Vol. III, McGraw-Hill (1950) pp. 27–52.
38. E. Reissner, On the equations for finite symmetrical deflections of this shells of revolution. In: D.C. Drucker (ed.), *Progress in Mechanics* (Prager Anniversary Volume). Macmillan, New York (1963) pp. 171–178.
39. E. Schwerin, Über Spannungen und Formänderungen kreisringförmiger Membranen. *Z. Tech. Phys.* 12 (1929) 651–659.
40. J.G. Simmonds, Rigorous expunction of Poisson's ratio from the Reissner-Meissner equations. *Int. J. Solids & Structures* 11 (1975) 1051–1056.

41. D. Steigmann, Proof of a conjecture in elastic membrane theory. *J. Appl. Mech.* 53 (1988) 955.
42. M. Stein and J.M. Hedgepeth, Analysis of partly wrinkled membranes, NASA TN D-813, July, 1961.
43. C.A. Stuart, Integral equations with decreasing nonlinearities and applications. *J. Differential Equations* 18 (1975) 202–217.
44. H.J. Weinitschke, Some mathematical problems in the nonlinear theory of elastic membranes, plates and shells. In: G. Fichera (ed.), *Trends in Applications of Pure Mathematics to Mechanics*. Pitman, London (1976) pp. 409–429.
45. H.J. Weinitschke, Stable and unstable axisymmetric solutions for membranes of revolution. *Appl. Mech. Rev.* 42, No. 11, Part 2 (1989) S289–S294.
46. H.J. Weinitschke, *Zur mathematischen Theorie der endlichen Verbiegung elastischer Platten*. Habilitationsschrift, Universität Hamburg (1965).
47. H.J. Weinitschke, Existenz- und Eindeutigkeitssätze für die Gleichungen der kreisförmigen Membran. *Meth. u. Verf. d. Math. Physik* 3 (1970) 117–139.
48. H.J. Weinitschke, On axisymmetric deformations of nonlinear elastic membranes. In: S. Nemat-Nasser (ed.), *Mechanics Today* 5 (E. Reissner Anniversary Volume). Pergamon Press, Oxford (1980) pp. 523–542.
49. H.J. Weinitschke, On finite displacements of circular elastic membranes. *Math. Meth. Appl. Sci.* 9 (1987) 76–98.